Summarizing Variation
Matrix Algebra

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Goals of the session

- Introduce summary statistics
- Learn basics of matrix algebra
- Reinforce basic OpenMx syntax
- Introduction to likelihood

Computing Mean

- Formula $\Sigma(x_i)/N$

Computing Mean

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Sum

Computing Mean

- Formula $\Sigma(x_i)/N$

Sum

Each individual score (the i refers to the i^{th} individual)

Computing Mean

- Formula $\Sigma(x_i)/N$

Sum

Sample size

Each individual score (the i refers to the i^{th} individual)

Computing Mean

- Formula $\Sigma(x_i)/N$
- Can compute with

Computing Mean

- Formula $\Sigma(x_i)/N$
- Can compute with
  - Pencil
  - Calculator
  - SAS
  - SPSS
  - Mx
  - R, others….

Means and matrices

- For all continuous traits we will have a mean
  - Let's say we have 100 traits (a lot!)
  - If we had to specify each mean separately we would have something like this:
    - M1 M2 M3 … M99 M100 (without "…"!)
  - This is where matrices are useful: they are boxes to hold numbers

Means and matrices cont’d

- So rather than specify all 100 means in OpenMx we can define a matrix to hold all of our means:

Means and matrices cont' d

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```
mxMatrix(type="Full", nrow=1, ncol=100,
free=TRUE, values=5, name="M")
```

Means and matrices cont' d

- So rather than specify all 100 means in OpenMx we can define a matrix to hold all of our means:

mxMatrix(type="Full", nrow=1, ncol=100, free=TRUE, values=5, name="M")

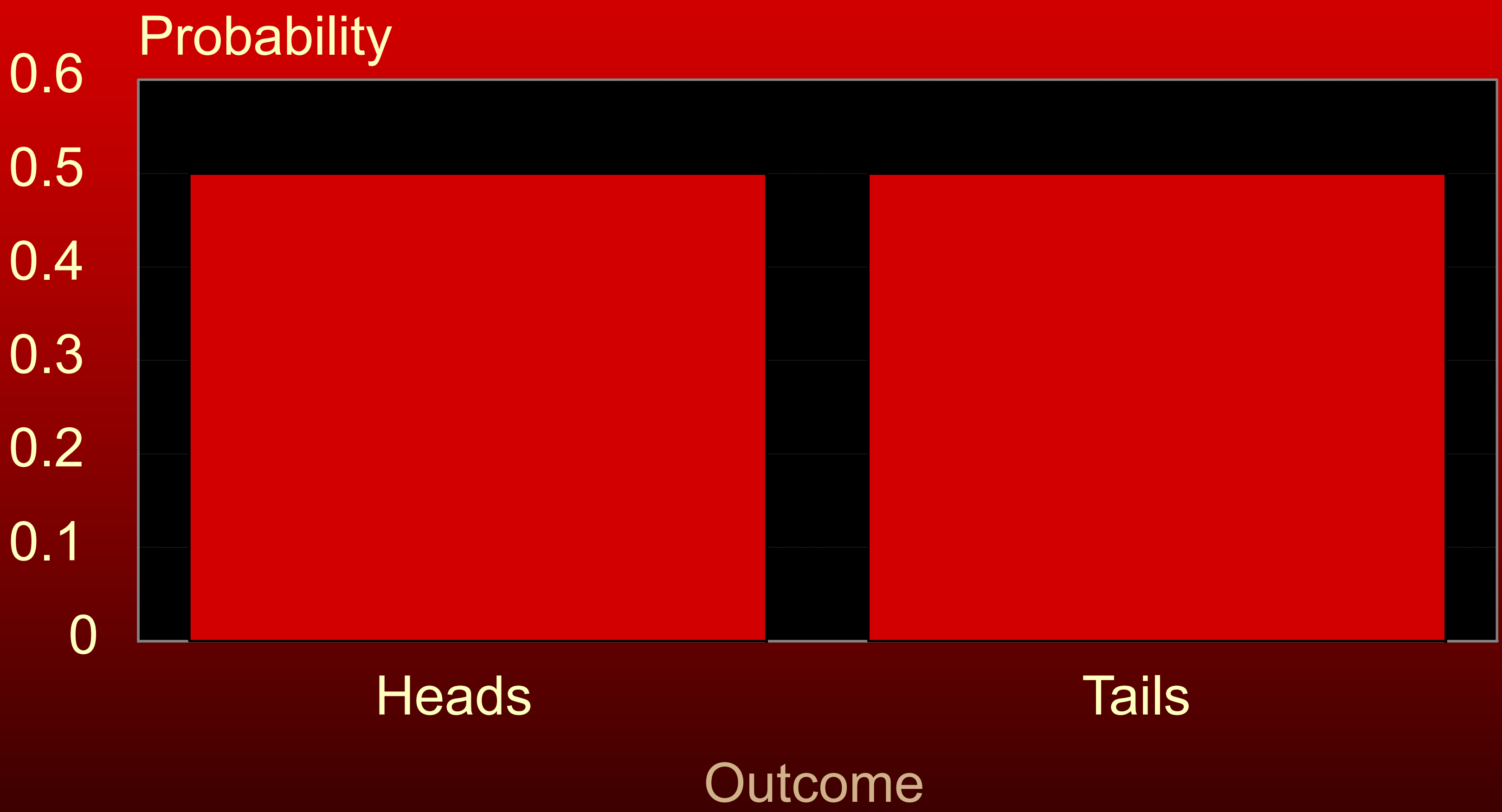
Quick note on specification of matrices in text:
Matrix M has dimension 1,100 meaning 1 row
And 100 columns – two mnemonics – like a procession
rows first then columns, or RC = Roman Catholic

To the Variance!

We'll start with coin tossing…

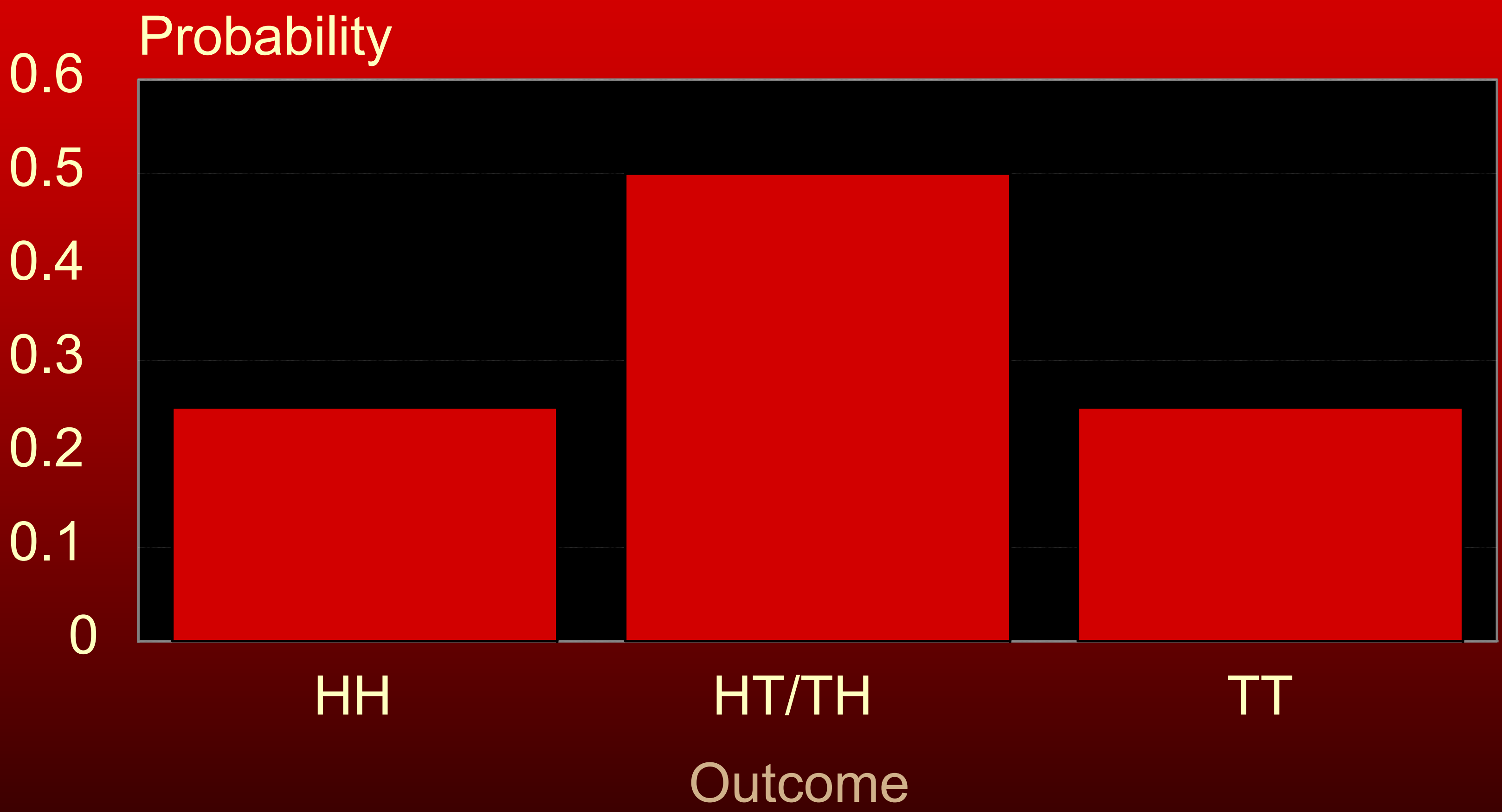
One Coin toss

2 outcomes



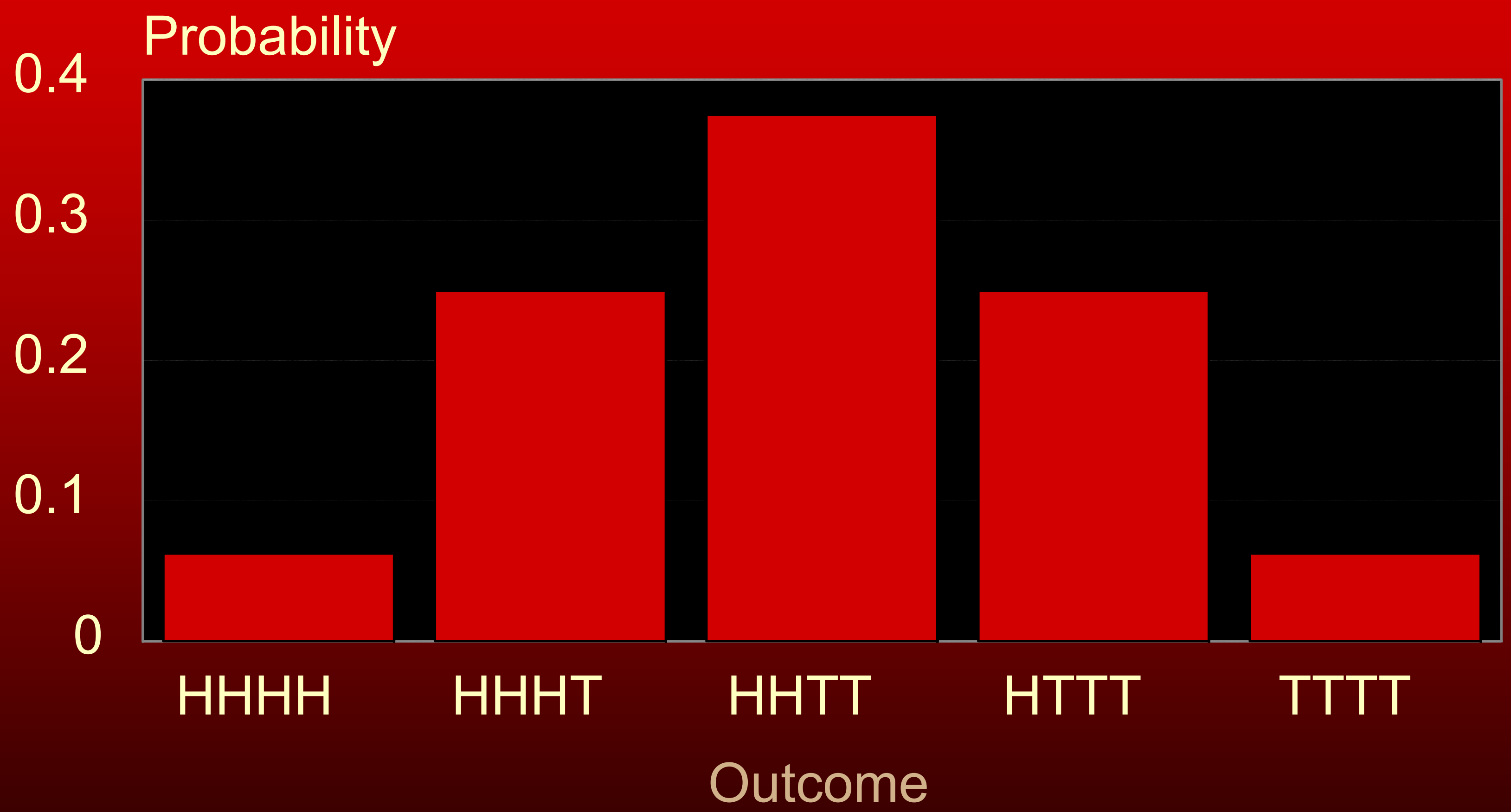
Two Coin toss

3 outcomes



Four Coin toss

5 outcomes

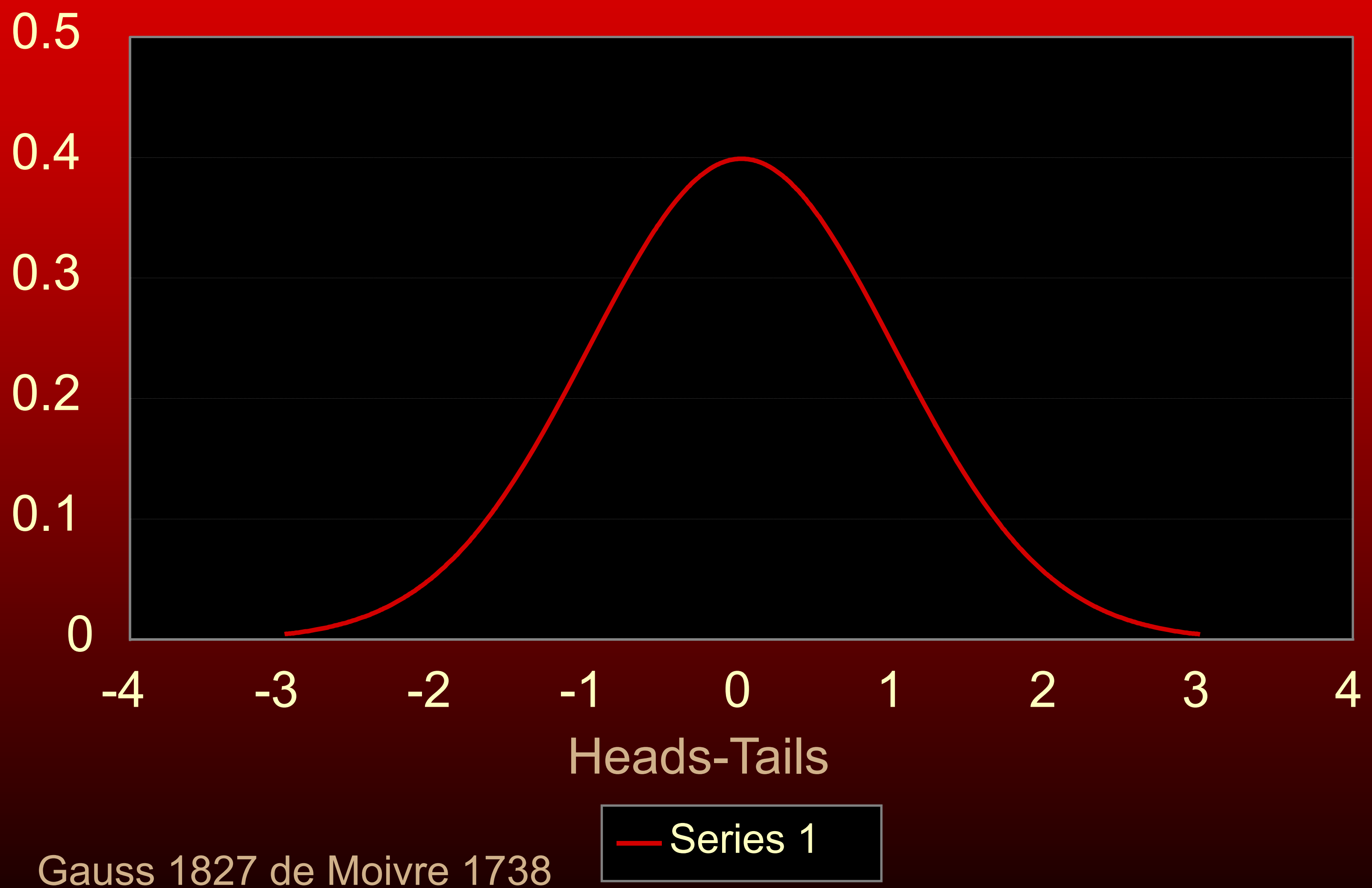


Ten Coin toss

11 outcomes

Fort Knox Toss

Infinite outcomes



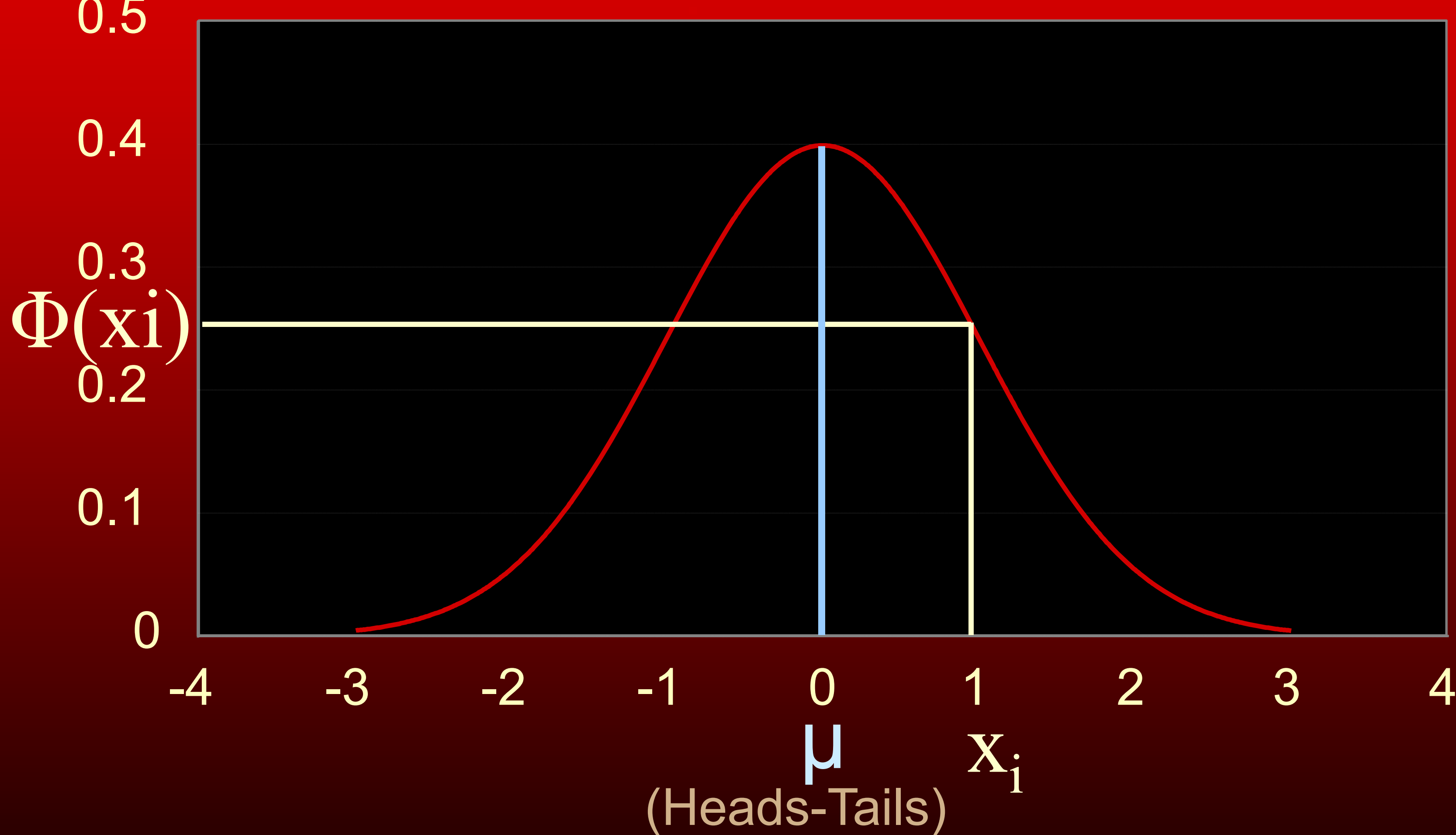
Gauss 1827 de Moivre 1738

Variance

- Measure of Spread
- Easily calculated
- Individual differences

Average squared deviation

Normal distribution



Gauss 1827 de Moivre 1738

Measuring Variation

Weighs & Means

- Absolute differences?
- Squared differences?
- Absolute cubed?
- Squared squared?

Measuring Variation

Ways & Means

- Squared differences

Fisher (1922) Squared has minimum variance under normal distribution

Variance calculation

1. Calculate mean

Variance calculation

1. Calculate mean
2. Calculate each squared deviation:

Variance calculation

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    1. Subtract the mean from each observations and square individually: $(x_i - \mu)^2$

Variance calculation

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    2. Sum up all the squared deviations

Variance calculation

1. Calculate mean
2. Calculate each squared deviation:
   1. Subtract the mean from each observations and square individually: $(x_i - \mu)^2$
   2. Sum up all the squared deviations
3. Divide sum of squared deviations by (N-1)

Your turn to calculate

Data

Subject	Weight in kg
1	80
2	70
3	75
4	85
5	90

1. Calculate mean
2. Calculate each deviation from the mean
3. Square each deviation individually
4. Sum the squared deviations
5. Divide by number of subjects - 1

Your turn to calculate

Data

Subject	Weight in kg
1	80
2	70
3	75
4	85
5	90

1. Calculate mean (80)
2. Calculate each deviation from the mean (0,-10,-5,5,10)
3. Square each deviation individually (0,100,25,25,100)
4. Sum the squared deviations (250)
5. Divide by number of subjects – 1 (250/(5-1))=(250/4)=62.5

Covariance

- Measure of association between two variables

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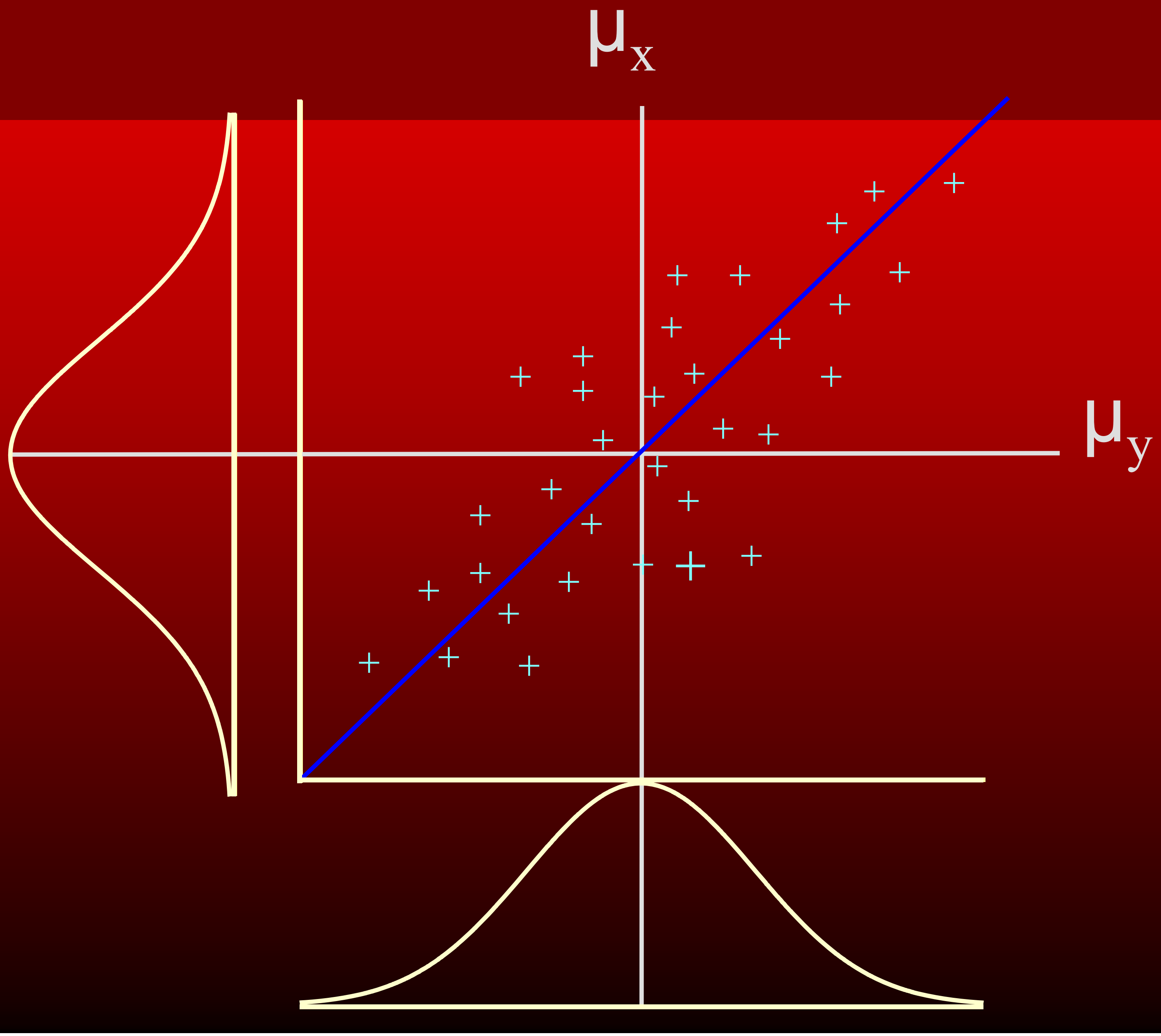
- Closely related to variance

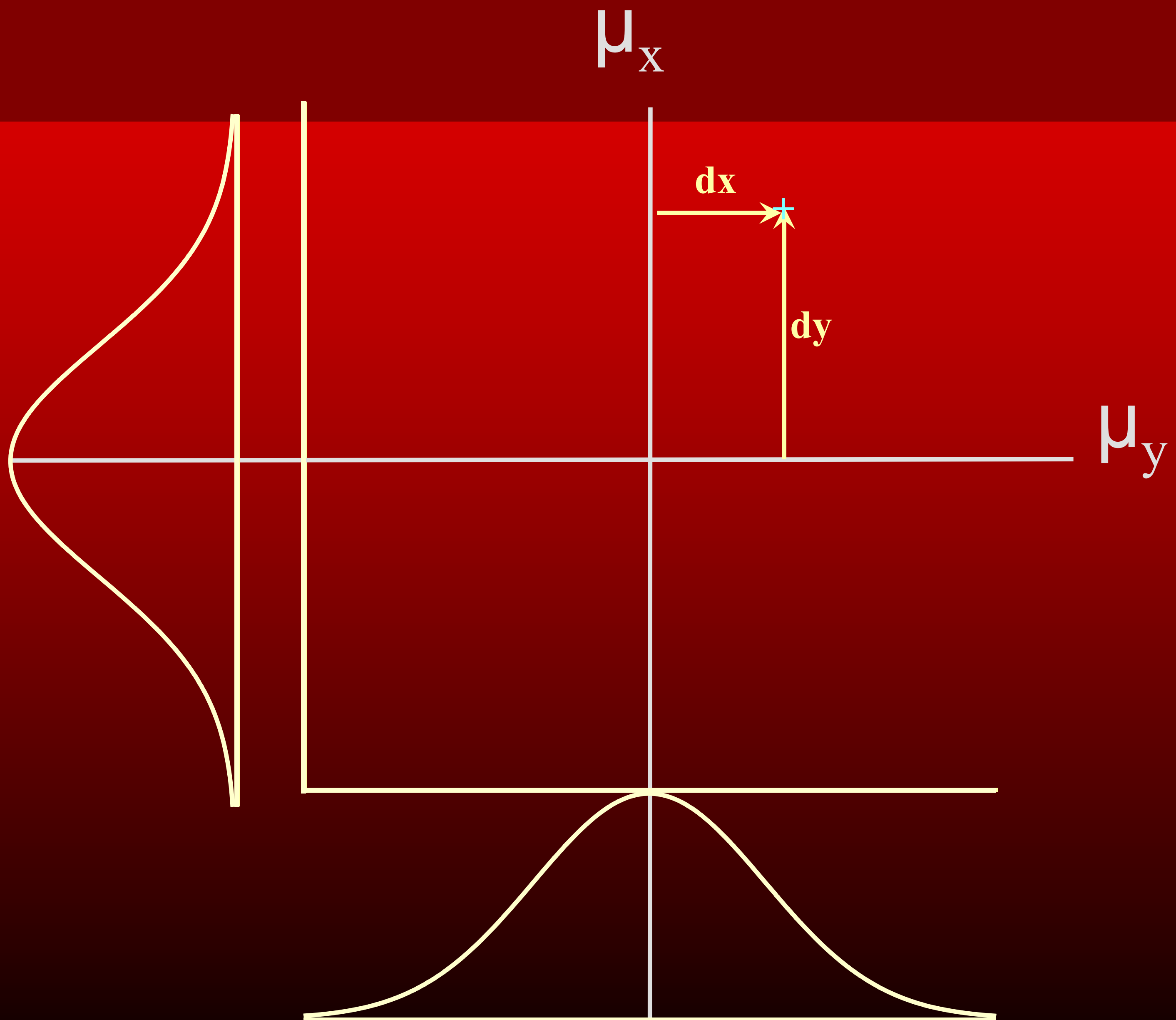
Covariance

- Measure of association between two variables
- Closely related to variance
- Useful to partition variance

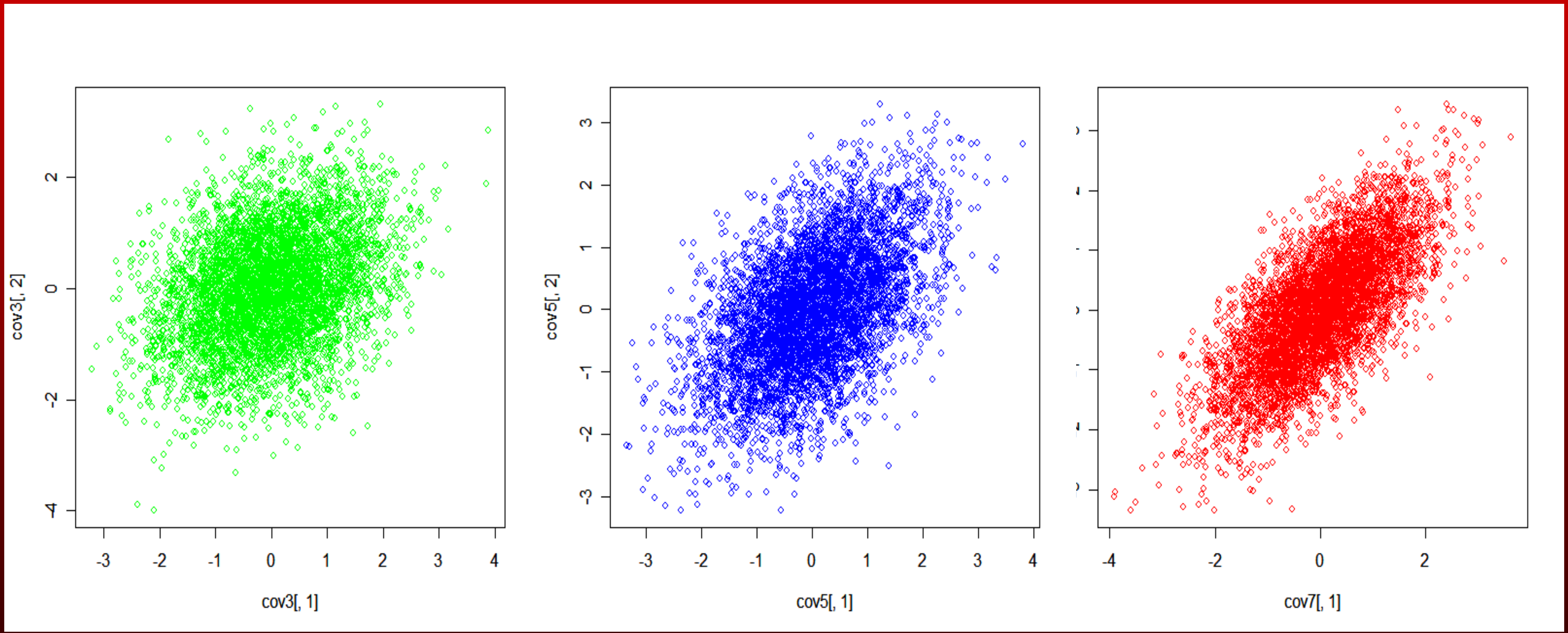
Deviations in two dimensions



Deviations in two dimensions



Different covariance



Covariance .3 Covariance .5 Covariance .7

Covariance calculation

1. Calculate mean for each variables: (μ_x, μ_y)

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   1. Subtract the mean for variable 1 from each observations of variable 1 $(x_i - \mu_x)$

Covariance calculation

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    2. Likewise for variable 2, making certain to keep the variables paired within subject $(y_i - \mu_y)$

Covariance calculation

1. Calculate mean for each variables: (μ_x, μ_y)
2. Calculate each deviation:
   1. Subtract the mean for variable 1 from each observations of variable 1 $(x_i - \mu_x)$
   2. Likewise for variable 2, making certain to keep the variables paired within subject $(y_i - \mu_y)$
3. Multiply the deviation for variable 1 and variable 2:

   $(x_i - \mu_x) * (y_i - \mu_y)$

Covariance calculation

1. Calculate mean for each variables: **(μ_x,μ_y)**
2. Calculate each deviation:
   1. Subtract the mean for variable 1 from each observations of variable 1 **($x_i - \mu_x$)**
   2. Likewise for variable 2, making certain to keep the variables paired within subject **($y_i - \mu_y$)**
3. Multiply the deviation for variable 1 and variable 2: **($x_i - \mu_x$) * ($y_i - \mu_y$)**
4. Sum up all the multiplied deviations: **$\Sigma(x_i - \mu_x) * (y_i - \mu_y)$**

Covariance calculation

1. Calculate mean for each variables: **(μ_x,μ_y)**
2. Calculate each deviation:
   1. Subtract the mean for variable 1 from each observations of variable 1 **$(x_i - \mu_x)$**
   2. Likewise for variable 2, making certain to keep the variables paired within subject **$(y_i - \mu_y)$**
3. Multiply the deviation for variable 1 and variable 2: **$(x_i - \mu_x) * (y_i - \mu_y)$**
4. Sum up all the multiplied deviations: **$\Sigma(x_i - \mu_x) * (y_i - \mu_y)$**
5. Divide sum of the multiplied deviations by (N-1):

   $\Sigma((x_i - \mu_x) * (y_i - \mu_y))/(N-1)$

Your turn to calculate

Data:

Subject	X	Y
1	80	180
2	70	175
3	75	170
4	85	190
5	90	185

1. Calculate means
2. Calculate each deviation:
   1. Subtract the mean X from each observations of X
   2. Likewise for Y
3. Multiply the deviation for X and Y pair-wise
4. Sum up all the multiplied deviation
5. Divide sum of the multiplied deviations by (N-1)

Your turn to calculate

Data:

Subject	X	Y
1	80	180
2	70	175
3	75	170
4	85	190
5	90	185

1. Calculate means: **(80,180)**
2. Calculate each deviation:
   1. Subtract the mean for variable 1 from each observations of variable 1 **(0,-10,-5,5,10)**
   2. Likewise for variable 2, making certain to keep the variables paired within subject **(0,-5,-10,10,5)**

Your turn to calculate

Data:

Subject	X	Y
1	80	180
2	70	175
3	75	170
4	85	190
5	90	185

3. Multiply the deviation for variable 1 and variable 2: **(0,-10,-5,5,10)*(0,-5,-10,10,5) (0,50,50,50,50)**
4. Sum up all the multiplied deviation
   0+50+50+50+50=200
5. Divide sum of the multiplied deviations by (N-1):
   200/4=50

Measuring Covariation

Covariance Formula

$$\sigma_{xy} = \frac{\sum (x_i - \mu_x) * (y_i - \mu_y)}{(N-1)}$$

Measuring Covariation

Covariance Formula

$$\sigma_{xy} = \frac{\Sigma(x_i - \mu_x) * (y_i - \mu_y)}{(N-1)}$$

Sum

Measuring Covariation

Covariance Formula

$$\sigma_{xy} = \frac{\sum (x_i - \mu_x) * (y_i - \mu_y)}{(N-1)}$$

Each individual score on trait x (the i refers to the i^{th} pair)

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Mean of x

Measuring Covariation

Covariance Formula

$$\sigma_{xy} = \frac{\sum (x_i - \mu_x) * (y_i - \mu_y)}{(N-1)}$$

Number of pairs

Measuring Covariation

Covariance Formula

$$\sigma_{xy} = \frac{\Sigma(x_i - \mu_x) * (y_i - \mu_y)}{(N-1)}$$

Each individual score on trait y (the i refers to the i^{th} pair)

Measuring Covariation

Covariance Formula

$$\sigma_{xy} = \frac{\sum (x_i - \mu_x) * (y_i - \mu_y)}{(N-1)}$$

Mean of y

Measuring Covariation

Covariance Formula

$$\sigma_{xy} = \frac{\Sigma(x_i - \mu_x) * (y_i - \mu_y)}{(N-1)}$$

Sum

Mean of x

Mean of y

Each individual score on trait x (the i refers to the i^{th} pair)

Number of pairs

Each individual score on trait x (the i refers to the i^{th} pair)

Variances, covariances, and matrices

For multiple traits we will have:

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a variance for each trait
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If we have 5 traits we have:

Variances, covariances, and matrices

For multiple traits we will have:
a variance for each trait
a covariance for each pair

If we have 5 traits we have:
5 variances (V1, V2, … V5)
10 covariances (CV1-2, CV1-3,…CV4-5

Variances, covariances, and matrices cont’d

- Just like means, we can put variances and covariances in a box
  - In fact this is rather convenient, because:
  - Cov(X,X) = Var(x) so the organization is natural

Variances, covariances, and matrices cont’d

	Trait 1	Trait 2	Trait 3	Trait 4
Trait 1	Var(T1)			
Trait 2				
Trait 3				
Trait 4				

Variances, covariances, and matrices cont’d

	Trait 1	Trait 2	Trait 3	Trait 4
Trait 1	Var(T1)			
Trait 2		Var(T2)		
Trait 3			Var(T3)	
Trait 4				Var(T4)

Variances, covariances, and matrices cont’d

	Trait 1	Trait 2	Trait 3	Trait 4
Trait 1		Cov(T1,T2)		
Trait 2				
Trait 3				
Trait 4				

Variances, covariances, and matrices cont’d

	Trait 1	Trait 2	Trait 3	Trait 4
Trait 1		Cov(T1,T2)	Cov(T1,T3)	Cov(T1,T4)
Trait 2	Cov(T2,T1)		Cov(T2,T3)	Cov(T2,T4)
Trait 3	Cov(T3,T1)	Cov(T3,T2)		Cov(T3,T4)
Trait 4	Cov(T4,T1)	Cov(T4,T2)	Cov(T4,T3)	

Variances, covariances, and matrices cont’d

	Trait 1	Trait 2	Trait 3	Trait 4
Trait 1		Cov(T1,T2)	Cov(T1,T3)	Cov(T1,T4)
Trait 2	Cov(T2,T1)		Cov(T2,T3)	Cov(T2,T4)
Trait 3	Cov(T3,T1)	Cov(T3,T2)		Cov(T3,T4)
Trait 4	Cov(T4,T1)	Cov(T4,T2)	Cov(T4,T3)	

Note Cov(T3,T1)=Cov(T1,T3)

Variances, covariances, and matrices cont’d

	Trait 1	Trait 2	Trait 3	Trait 4
Trait 1	Var(T1)	Cov(T1,T2)	Cov(T1,T3)	Cov(T1,T4)
Trait 2	Cov(T2,T1)	Var(T2)	Cov(T2,T3)	Cov(T2,T4)
Trait 3	Cov(T3,T1)	Cov(T3,T2)	Var(T3)	Cov(T3,T4)
Trait 4	Cov(T4,T1)	Cov(T4,T2)	Cov(T4,T3)	Var(T4)

Variances, covariances, and matrices in Mx

A variance covariance matrix is symmetric, which means the elements above and below the diagonal are identical

Variances, covariances, and matrices in Mx

A variance covariance matrix is symmetric, which means the elements above and below the diagonal are identical

$$\begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix}$$

Variances, covariances, and matrices in Mx

A variance covariance matrix is symmetric, which means the elements above and below the diagonal are identical

A	B	C	D
B	E	F	G
C	F	H	I
D	G	I	J

The Diagonal

Variances, covariances, and matrices in Mx

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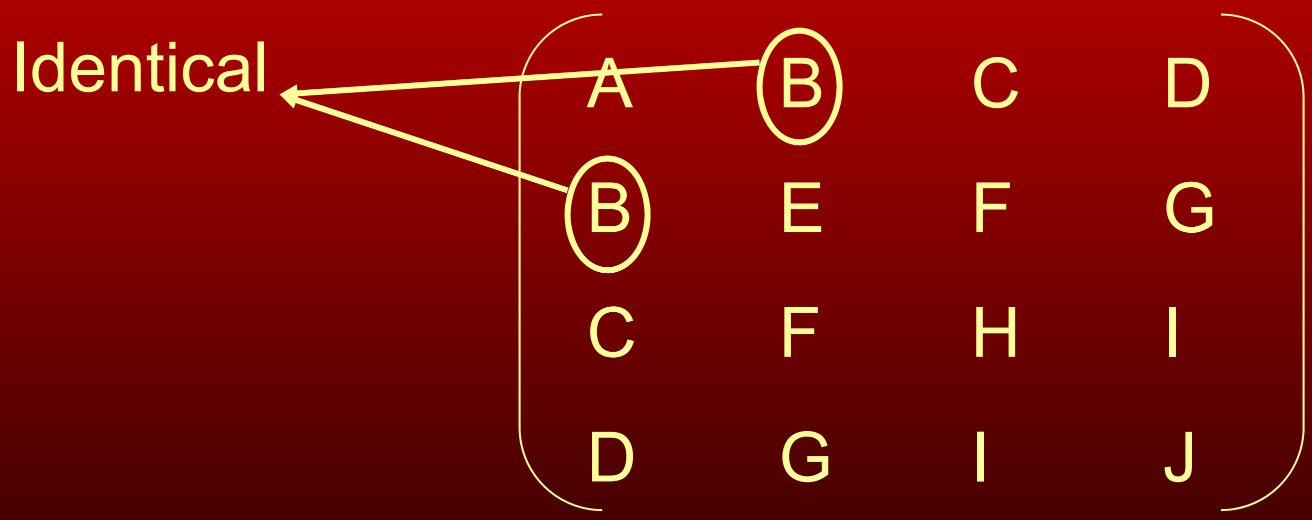

Variances, covariances, and matrices in Mx

A variance covariance matrix is symmetric, which means the elements above and below the diagonal are identical

Identical

Element 1,2 =
Element 2,1

A	B	C	D
B	E	F	G
C	F	H	I
D	G	I	J

Variances, covariances, and matrices in Mx

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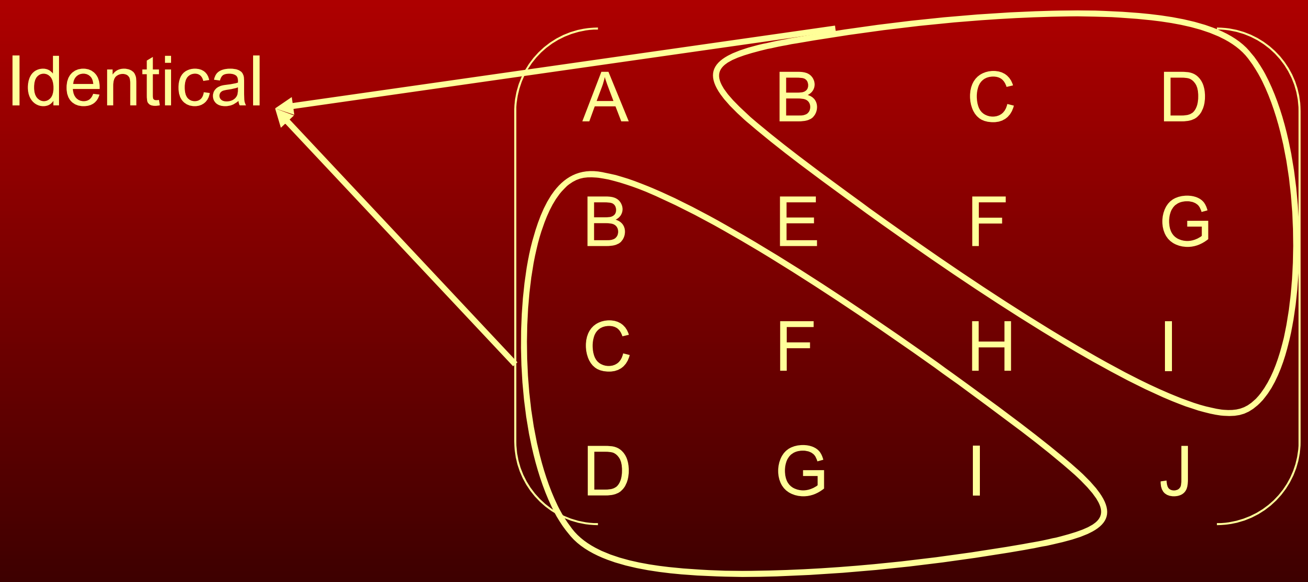

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Considered another way, a symmetric matrix's **transpose** is equal to itself.

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Transpose is exchanging rows and columns

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Transpose is exchanging rows and columns e.g.:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

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Transpose is exchanging rows and columns e.g.:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad A' = \text{ or } A^T \begin{pmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{pmatrix}$$

Variances, covariances, and matrices in Mx

A variance covariance matrix is symmetric, which means the elements above and below the diagonal are identical:

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A' = or A^T

$$\begin{pmatrix} 1 & \\ 2 & \\ 3 & \end{pmatrix}$$

Variances, covariances, and matrices in Mx

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$$A' = \text{ or } A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Variances, covariances, and matrices in Mx

OpenMx uses mxFIMLObjective to calculate the variance-covariance matrix for the data (let's stick with matrix V)

mxMatrix(type="Sym", nrow=2, ncol=2, free=TRUE, values=.5, name="V")

type specifies what kind of matrix – symmetric

nrow specifies the number of rows

ncol specifies the number of columns

free means that OpenMx will estimate

How else can we get a symmetric matrix?

We can use matrix algebra and another matrix type.

How else can we get a symmetric matrix?

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We will be using matrix multiplication and lower matrices

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We can use matrix algebra and another matrix type.

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Let’s begin with matrix multiplication

Matrix multiplication

Surprisingly, everyone in this room has completed matrix multiplication

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Let's extend this to a simple case: Two 2x2 matrices

Matrix multiplication

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Let's extend this to a simple case: Two 2x2 matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \%*\% \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

Matrix multiplication example

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Matrix multiplication example

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \%*\% \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

2 x 2 2 x 2

RULE: The number of columns in the 1st matrix must equal the number of rows in the 2nd matrix.

Matrix multiplication example

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Step 1: Determine the size of the box (matrix) needed for the answer

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$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \%{*}\% \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

Step 1: Determine the size of the box (matrix) needed for the answer

- The resulting matrix will have dimension: the number of rows of the 1^{st} matrix by the number of columns of the 2^{nd} matrix

Matrix multiplication example

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \%{*}\% \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \left(\phantom{\begin{matrix} X & X \\ X & X \end{matrix}} \right.$$

2 x 2 2 x 2 2

Step 1: Determine the size of the box (matrix) needed for the answer

- The resulting matrix will have dimension: **the number of rows of the 1st matrix** by the number of columns of the 2nd matrix

Matrix multiplication example

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \%*\% \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

2 x 2 | 2 x 2 | 2 x 2

Step 1: Determine the size of the box (matrix) needed for the answer

- The resulting matrix will have dimension: the number of rows of the 1st matrix by **the number of columns of the 2nd matrix**

Matrix multiplication example

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{\%*\%} \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

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2 x 2 2 x 2 2 x 2

Step 2: Calculate each element of the resulting matrix.

Matrix multiplication example

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \%*\% \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} \bigcirc & \\ & \end{pmatrix}$$

2 x 2 2 x 2 2 x 2

Step 2: Calculate each element of the resulting matrix.

There are 4 elements in our new matrix: **1,1**

Matrix multiplication example

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \%*\% \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} & \bigcirc \\ & \end{pmatrix}$$

2 x 2   2 x 2   2 x 2

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There are 4 elements in our new matrix: 1,1; **1,2;**

Matrix multiplication example

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2 x 2 2 x 2 2 x 2

Step 2: Calculate each element of the resulting matrix.

There are 4 elements in our new matrix: 1,1; 1,2; **2,1**;

Matrix multiplication example

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{\%*\%} \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} & \\ & \bigcirc \end{pmatrix}$$

2 x 2 2 x 2 2 x 2

Step 2: Calculate each element of the resulting matrix.

There are 4 elements in our new matrix: 1,1; 1,2; 2,1; **2,2**

Matrix multiplication example

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \%*\% \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

2 x 2   2 x 2   2 x 2

Step 2: Calculate each element of the resulting matrix. Each element is the cross-product of the corresponding row of the 1st matrix and the corresponding column of the 2nd matrix (e.g. element 2,1 will be the cross product of the 2nd row of matrix 1 and the 1st column of matrix 2)

What's a cross-product?

- Used in the context of vector multiplication:
- We have 2 vectors of the same length (same number of elements)
- The cross-product (x) is the sum of the products of the elements, e.g.:
    - Vector 1 = { a b c } Vector 2 = { d e f }
    - V1 x V2 = a*d +b*e +c*f

Matrix multiplication example

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \%{*}\% \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} \bigcirc & \\ & \end{pmatrix}$$

2 x 2   2 x 2   2 x 2

Step 2, element 1,1: cross-product row1 matrix1 with column1 matrix2

{ A B } x { E G } = (A*E + B*G)

Matrix multiplication example

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \%*\% \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} AE + BG & \bigcirc \\ & \end{pmatrix}$$

2 x 2   2 x 2   2 x 2

Step 2, element 1,2: cross-product row1 matrix1 with column2 matrix2

{ A B } x { F H } = (A*F + B*H)

Matrix multiplication example

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \%{*}\% \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} AE + BG & AF + BH \\ \bigcirc & \end{pmatrix}$$

2 x 2   2 x 2   2 x 2

Step 2, element 2,1: cross-product row2 matrix1 with column1 matrix2

{ C D } x { E G } = (C*E + D*G)

Matrix multiplication example

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \%*\% \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} AE + BG & AF + BH \\ CE + DG & \bigcirc \end{pmatrix}$$

2 x 2   2 x 2   2 x 2

Step 2, element 2,2: cross-product row2 matrix1 with column2 matrix2

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Matrix multiplication exercises

$L = \begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix}$ $A = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ $B = \begin{pmatrix} 4 & 7 & -4 \end{pmatrix}$ $C = \begin{pmatrix} -4 & 2 \\ -3 & 2 \\ 2 & -2 \end{pmatrix}$

Calculate:

1. L%*% t(L) (L times L transpose)
2. B%*%A
3. A%*%B
4. C%*%A

Matrix multiplication answers

1 $\begin{pmatrix} 4 & 2 \\ 2 & 17 \end{pmatrix}$

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Matrix multiplication answers

1 $\begin{pmatrix} 4 & 2 \\ 2 & 17 \end{pmatrix}$

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3 $\begin{pmatrix} 4 & 7 & 4 \\ 8 & 14 & 8 \\ 20 & 35 & 20 \end{pmatrix}$

4 Incomputable

What have we learned?

- Matrix multiplication is not commutative:
  - E.g. A*B ≠ B*A
- The product of a lower matrix and its transpose is symmetric
- Not all matrices were made to multiply with one another

Correlation

- Standardized covariance
- Calculate by taking the covariance and dividing by the square root of the product of the variances
- Lies between -1 and 1

$$r_{xy} = \frac{\sigma_{xy}}{\sqrt{\sigma_x^2 * \sigma_y^2}}$$

Calculating Correlation from data

Data:

Subject	X	Y
1	80	180
2	70	175
3	75	170
4	85	190
5	90	185

1. Calculate mean for X and Y
2. Calculate variance for X and Y
3. Calculate covariance for X and Y
4. Correlation formula is $cov_{xy}/(\sqrt{(var_x * var_y)})$
5. Variance of X is 62.5, and Variance of Y is 62.5, covariance(X,Y) is 50

Calculating Correlation from data

Data:

Subject	X	Y
1	80	180
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3	75	170
4	85	190
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1. Correlation formula is $cov_{xy}/(\sqrt{(var_x * var_y)})$
2. Variance of X is 62.5, and Variance of Y is 62.5, covariance(X,Y) is 50
3. Answer: $50/(\sqrt{(62.5*62.5)})=.8$

Standardization

- We can standardize an entire variance covariance matrix and turn it into a correlation matrix

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- What will we need?

Covariance(X,Y)

$$r_{xy} = \frac{\sigma_{xy}}{\sqrt{\sigma_x^2 * \sigma_y^2}}$$

Variance(X)

Variance(Y)

Standardization back to boxes

- We'll start with our Variance Covariance matrix:

$$V = \begin{pmatrix} V_1 & CV_{12} & CV_{13} \\ CV_{21} & V_2 & CV_{23} \\ CV_{31} & CV_{32} & V_3 \end{pmatrix}$$

Standardization back to boxes

- We' ll start with our Variance Covariance matrix:

$$V = \begin{pmatrix} V_1 & CV_{12} & CV_{13} \\ CV_{21} & V_2 & CV_{23} \\ CV_{31} & CV_{32} & V_3 \end{pmatrix}$$

•We' ll use 4 operators:

•Dot product (*)

•Inverse (solve)

•Square root (\sqrt)

•Pre-multiply and post-multiply by the transpose (%&%)

Standardization formula

Formula to standardize V:

solve (sqrt (I*V)) %&% V

Extracts the variances

The dot product does element by element multiplication, so the dimensions of the two matrices must be equal, resulting in a matrix of the same size

Standardization formula

Formula to standardize V:

solve (sqrt (I*V)) %&% V

Extracts the variances

In this case I is an identity matrix of same size as V (3,3):

1	0	0
0	1	0
0	0	1

Identity matrices have the property of returning the same matrix as the original in standard multiplication

Standardization formula

Formula to standardize V:

solve (sqrt (I*V)) %&% V

Extracts the variances

So I*V =

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} V_1 & CV_{12} & CV_{13} \\ CV_{21} & V_2 & CV_{23} \\ CV_{31} & CV_{32} & V_3 \end{pmatrix} = \begin{pmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & V_3 \end{pmatrix}$$

Standardization formula

Formula to standardize V:

solve (sqrt (I*V)) %&% V

Square root

Take the square root of each element

$$\begin{pmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & V_3 \end{pmatrix} \rightarrow \begin{pmatrix} \sqrt{V_1} & 0 & 0 \\ 0 & \sqrt{V_2} & 0 \\ 0 & 0 & \sqrt{V_3} \end{pmatrix}$$

Standardization formula

Formula to standardize V:

solve (sqrt (I*V)) %&% V

Inverts the matrix

The inverse of matrix B is the matrix that when multiplied by B yields the identity matrix:

B %*% B^{-1} = B^{-1} %*%B = I

Warning: not all matrices have an inverse. A matrix with no inverse is a **singular** matrix. A zero matrix is a good example.

Standardization formula

Formula to standardize V:

solve (sqrt (I*V)) %&% V

Inverts the matrix

Let's call this K:

$$\begin{pmatrix} \sqrt{V_1} & 0 & 0 \\ 0 & \sqrt{V_2} & 0 \\ 0 & 0 & \sqrt{V_3} \end{pmatrix}^{-1} \bullet= \begin{pmatrix} 1/\sqrt{V_1} & 0 & 0 \\ 0 & 1/\sqrt{V_2} & 0 \\ 0 & 0 & 1/\sqrt{V_3} \end{pmatrix}$$

Operator roundup

Operator	Function	Rule
%*%	Matrix multiplication	$C_1=R_2$
*	Dot product	$R_1=R_2$ & $C_1=C_2$
/	Element-by-element division	$R_1=R_2$ & $C_1=C_2$
solve	Inverse	non-singular
+	Element-by-element addition	$R_1=R_2$ & $C_1=C_2$
-	Element-by-element subtraction	$R_1=R_2$ & $C_1=C_2$
%&%	Pre- and post-multiplication	$C_1=R_2=C_2$

Full list can be found:

http://openmx.psyc.virginia.edu/wiki/matrix-operators-and-functions