Measurement invariance
in the linear factor model: practical

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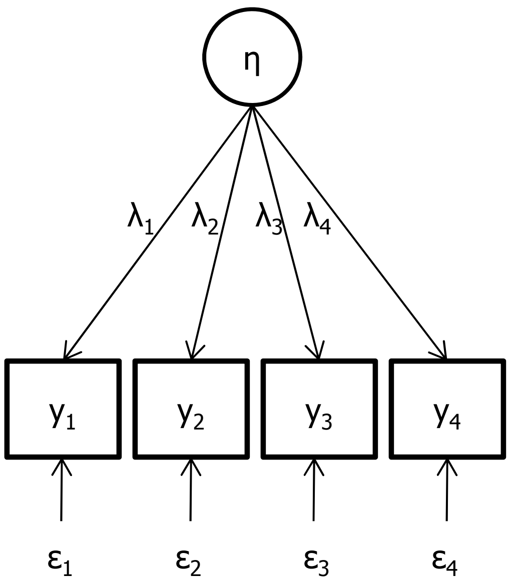
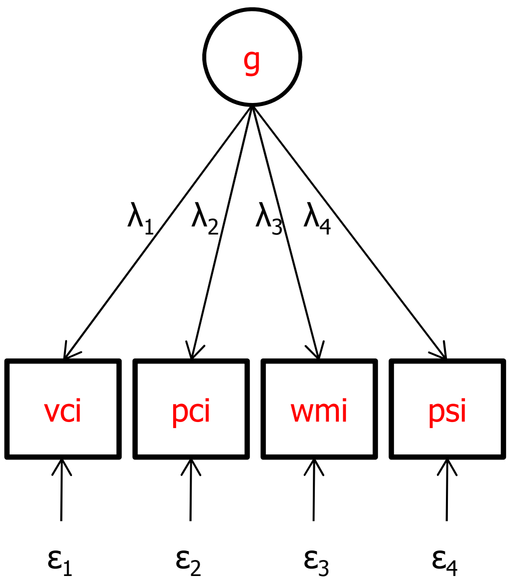
↓

model that relates a continuous latent variable to continuous indicators

Linear factor model



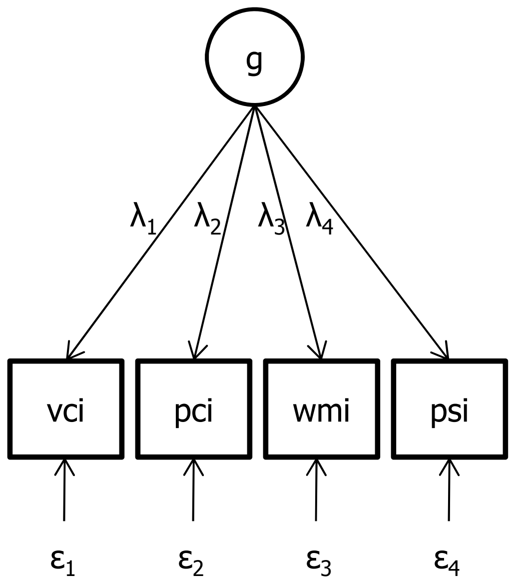
Linear factor model



IQ test (e.g. WAIS):

vci -- Verbal Comprehension Index
poi -- Perceptual Organization Index
wmi -- Working Memory Index
psi -- Processing Speed Index

Linear factor model

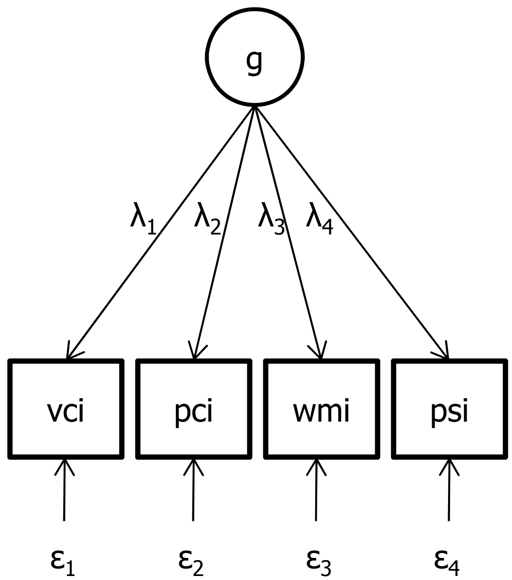


Do males, on average ,score differently than the females?

Men score significantly higher: MANOVA -> $p<.01$

Does this imply that women have a lower level of g?

Linear factor model



Do males, on average ,score differently than the females?

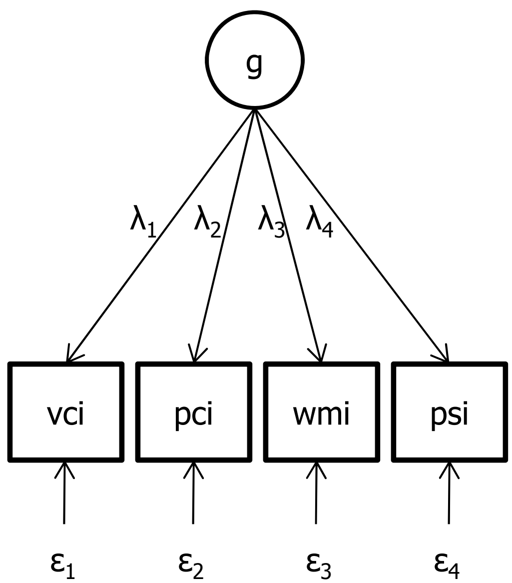
Men score significantly higher: MANOVA -> $p<.01$

Does this imply that women have a lower level of g?

Not necessarily.

It depends on whether the test measures the same construct in males as it does in females.

Linear factor model



Conditional distributions in 2 groups (conditional on a given value of η (η^*)):

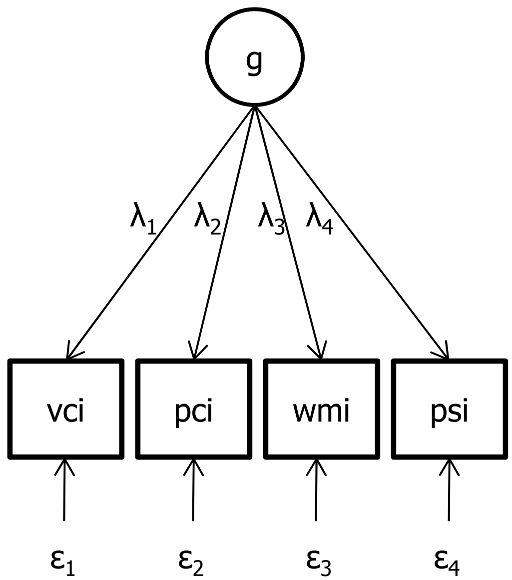
$y_{1i} | \eta^* \sim N(T_1 + \Lambda_1 \eta^*, \Theta_1)$

$y_{2i} | \eta^* \sim N(T_2 + \Lambda_2 \eta^*, \Theta_2)$

MI requires these distributions to be equal.

Linear factor model

$\boldsymbol{\Sigma = \Lambda \Psi \Lambda^t + \Theta}$



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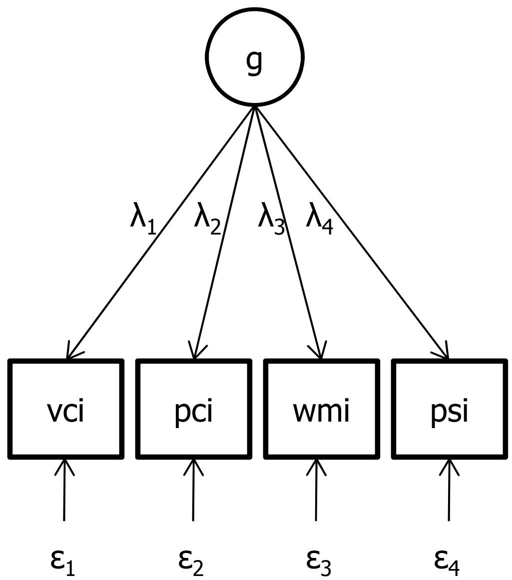

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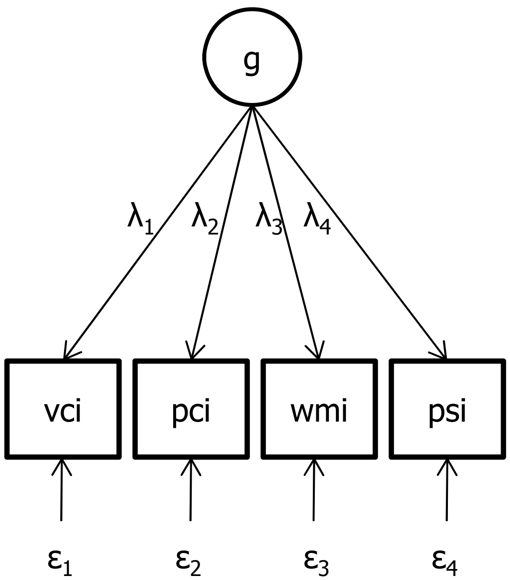

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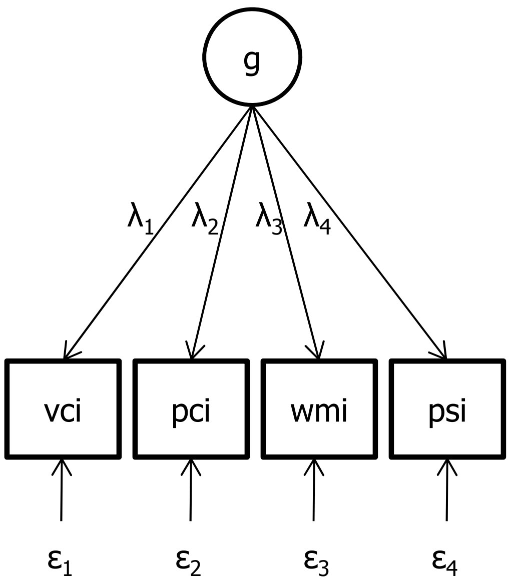

Conditional distributions in 2 groups (conditional on a given value of η (η*)):

$\mathbf{y_{1i} | \eta^* \sim N} (\mathbf{T_1 + \Lambda_1 \eta^*}, \Theta_1)$

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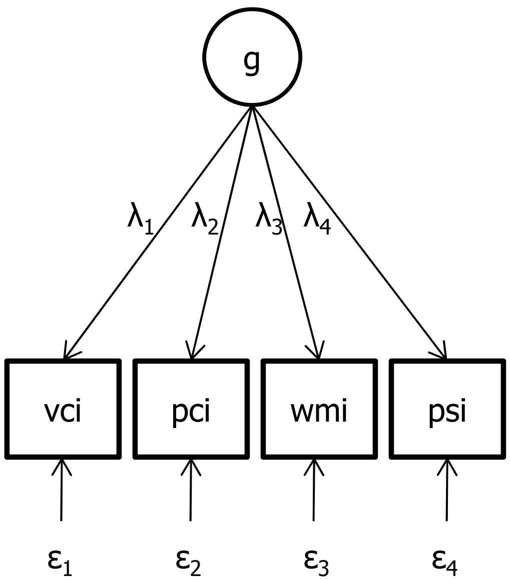

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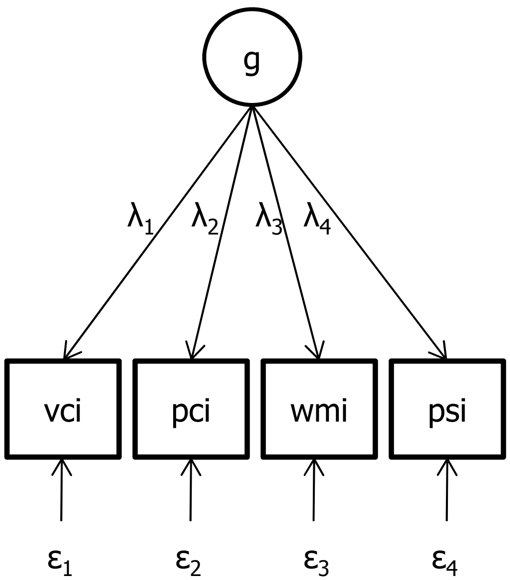

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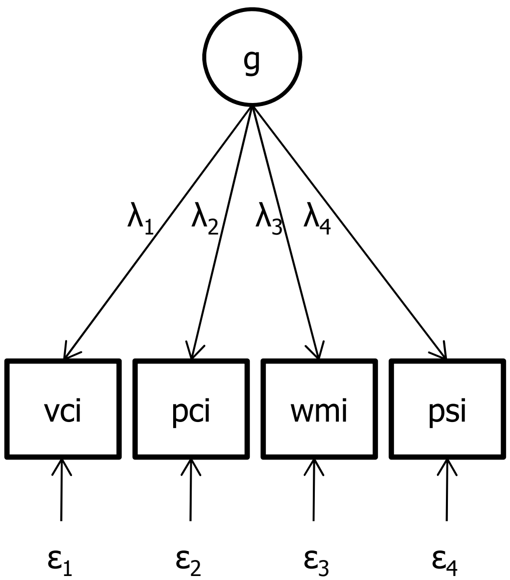
MI requires these distributions to be equal.

Linear factor model

$\boldsymbol{\Sigma} = \boldsymbol{\Lambda}\ \mathbf{0}\ \boldsymbol{\Lambda}^t + \boldsymbol{\Theta}$
$= \boldsymbol{\Theta}$



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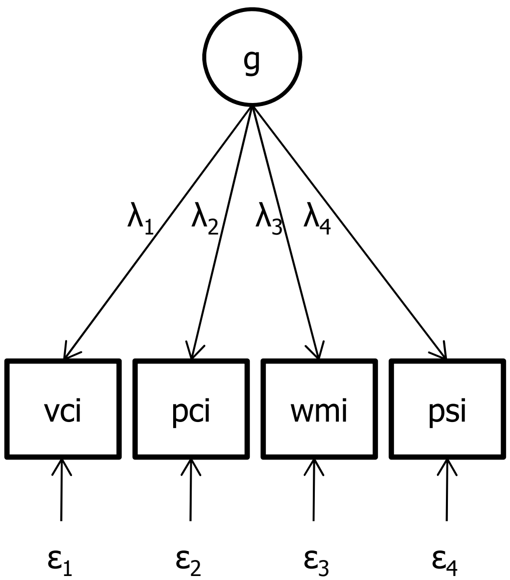
MI requires these distributions to be equal.

Linear factor model

$$\boldsymbol{\Sigma = \Lambda\, 0\, \Lambda^t + \Theta}$$
$$\boldsymbol{= \Theta}$$

$$\boldsymbol{E\,[y\,|\,\eta^*] = T + \Lambda\,\eta^*}$$



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$\boldsymbol{y_{1i}\,|\,\eta^* \sim N\,(T_1 + \Lambda_1\,\eta^*, \Theta_1)}$

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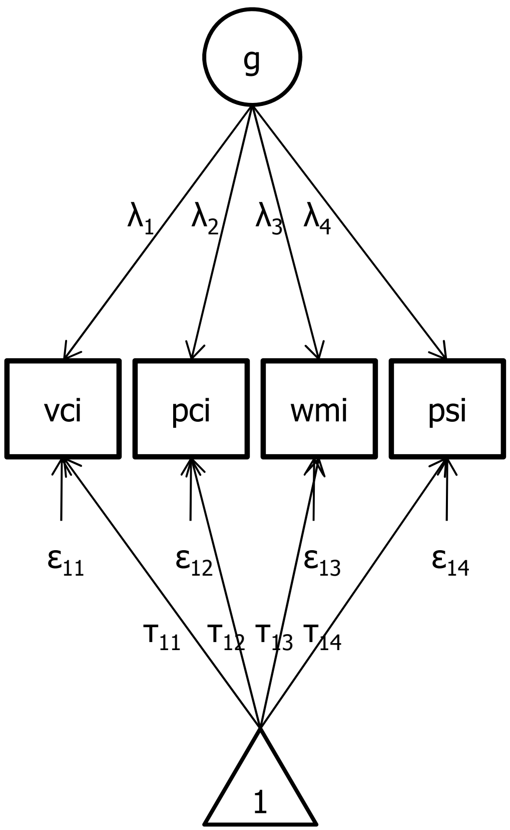
Linear factor model

$\Sigma = \Lambda \, 0 \, \Lambda^t + \Theta$
$= \Theta$

$E[y \mid \eta^*] = T + \Lambda \eta^*$



Conditional distributions in 2 groups (conditional on a given value of η (η*)):

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Linear factor model

$\Sigma = \Lambda\ 0\ \Lambda^t + \Theta$
$= \Theta$

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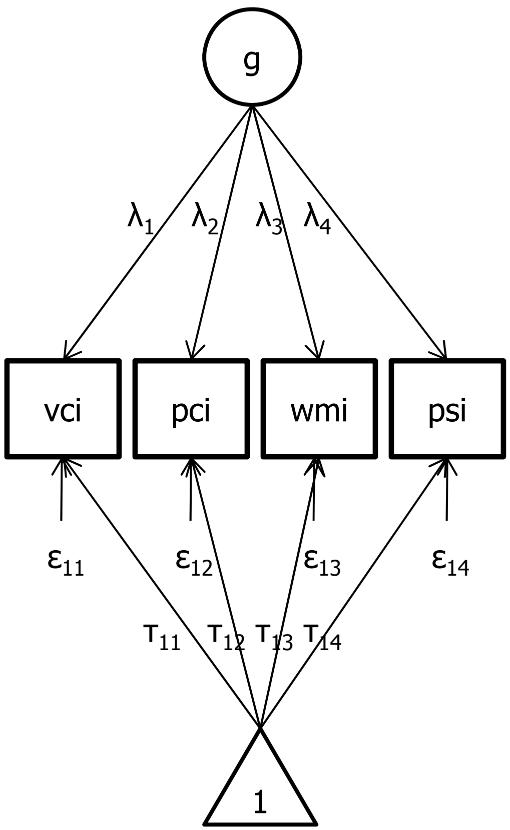

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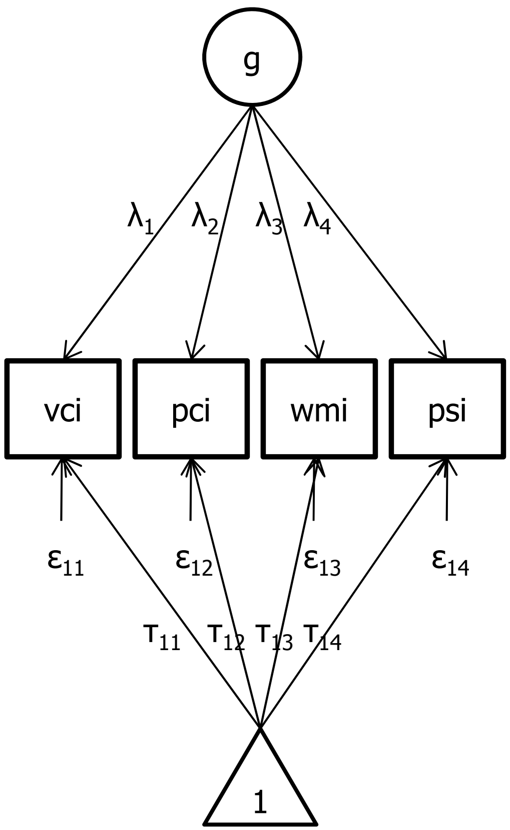

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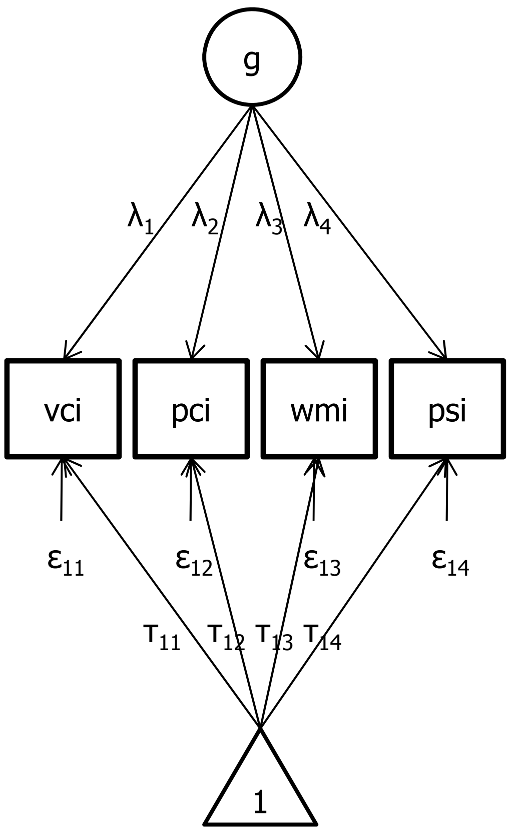

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MI requires these distributions to be equal.

This is the case if and only if:

$T_1 = T_2$

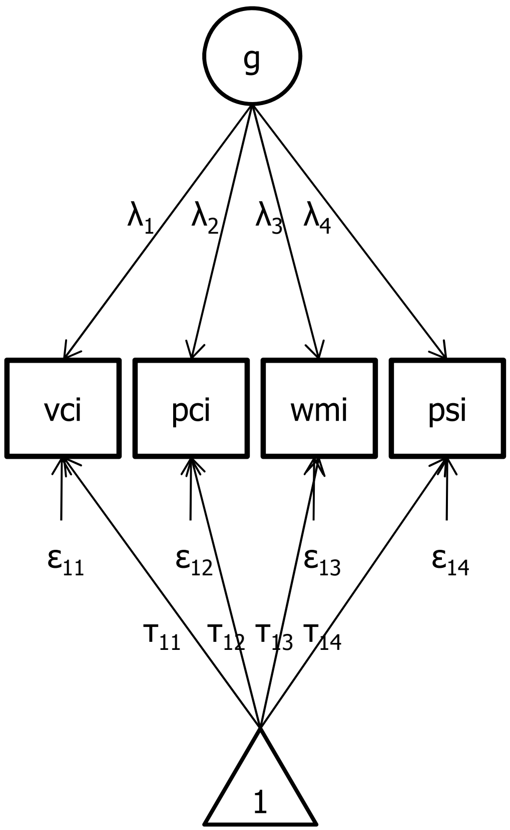
$\Lambda_1 = \Lambda_2$

$\Theta_1 = \Theta_2$

Linear factor model

$$\Sigma = \Lambda\, 0\, \Lambda^t + \Theta$$
$$= \Theta$$

$$E\,[y \mid \eta^*] = T + \Lambda\, \eta^*$$



Conditional distributions in 2 groups (conditional on a given value of η (η*)):

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MI requires these distributions to be equal.

This is the case if and only if:

$T_1 = T_2$

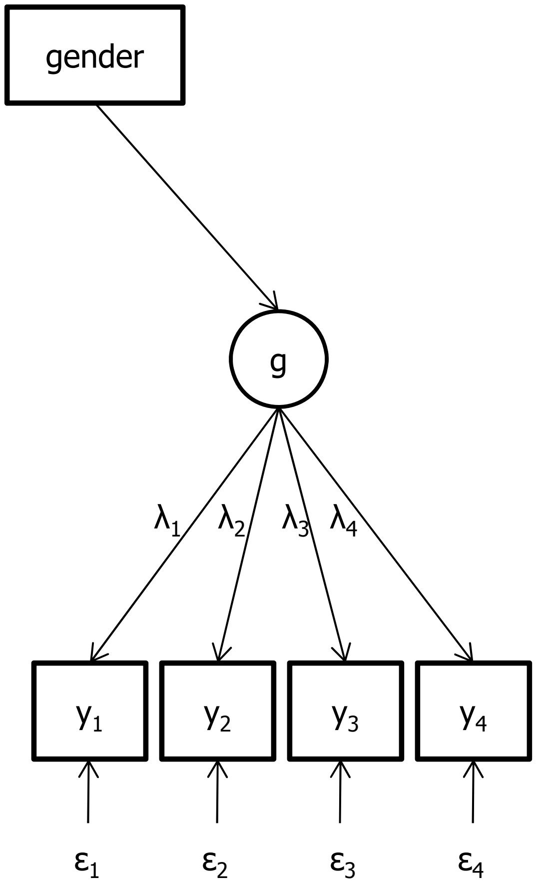
$\Lambda_1 = \Lambda_2$

$\Theta_1 = \Theta_2$

The test is MI with respect to group if the observed group differences in summary statistics (means and covariance matrix) are attributable to differences in the means and variance of the latent trait or common factor (Ψ_k and α_k).

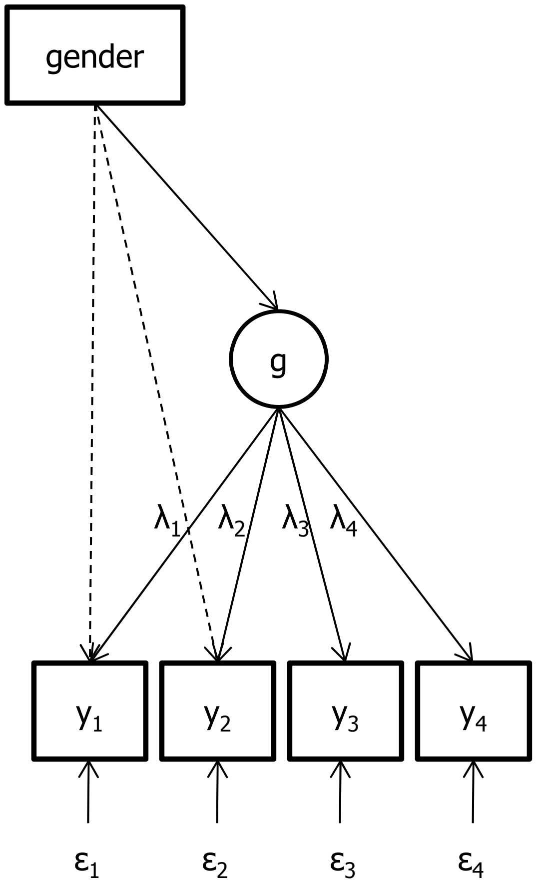
-> if the test measures the same latent variable in the two groups, then that latent variable should be the only source of differences between the groups.

Linear factor model



Measurement invariance

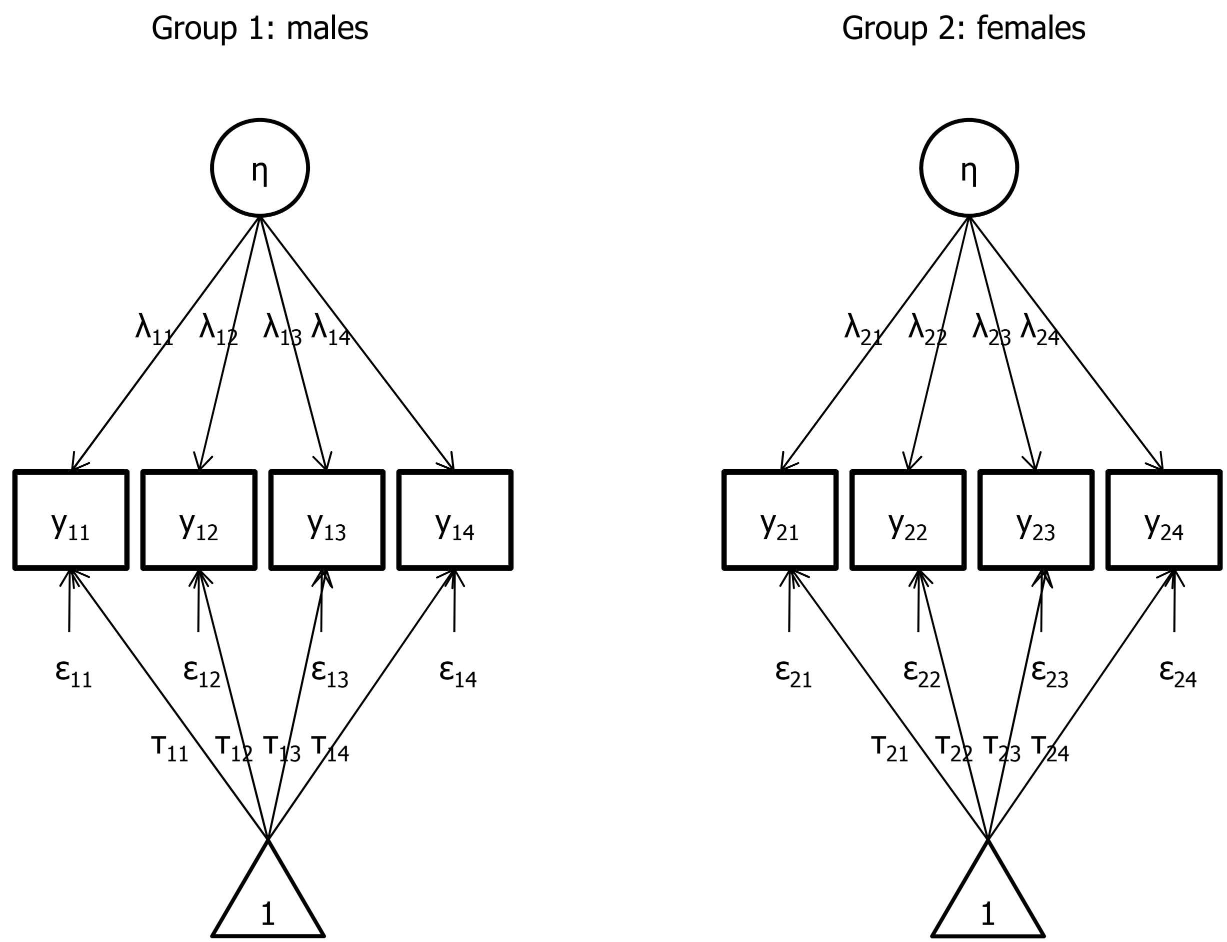
Linear factor model



Lack of measurement invariance

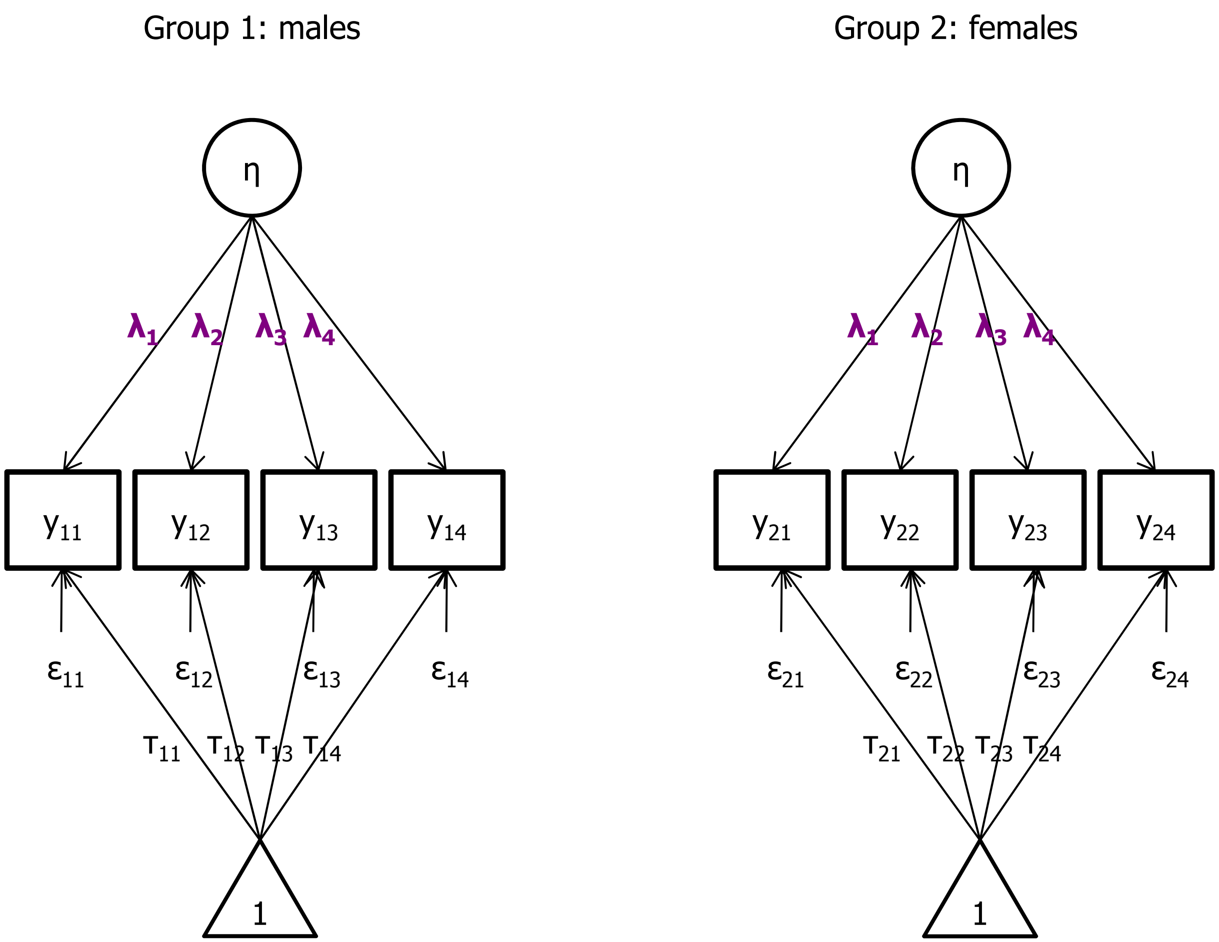
Establishing MI: testing a number of increasingly restrictive models

MODEL 1: Configural invariance -> in the 2 groups the same indicators load on the same factors
(i.e., the pattern or configuration of Λ and Θ are the same over groups)



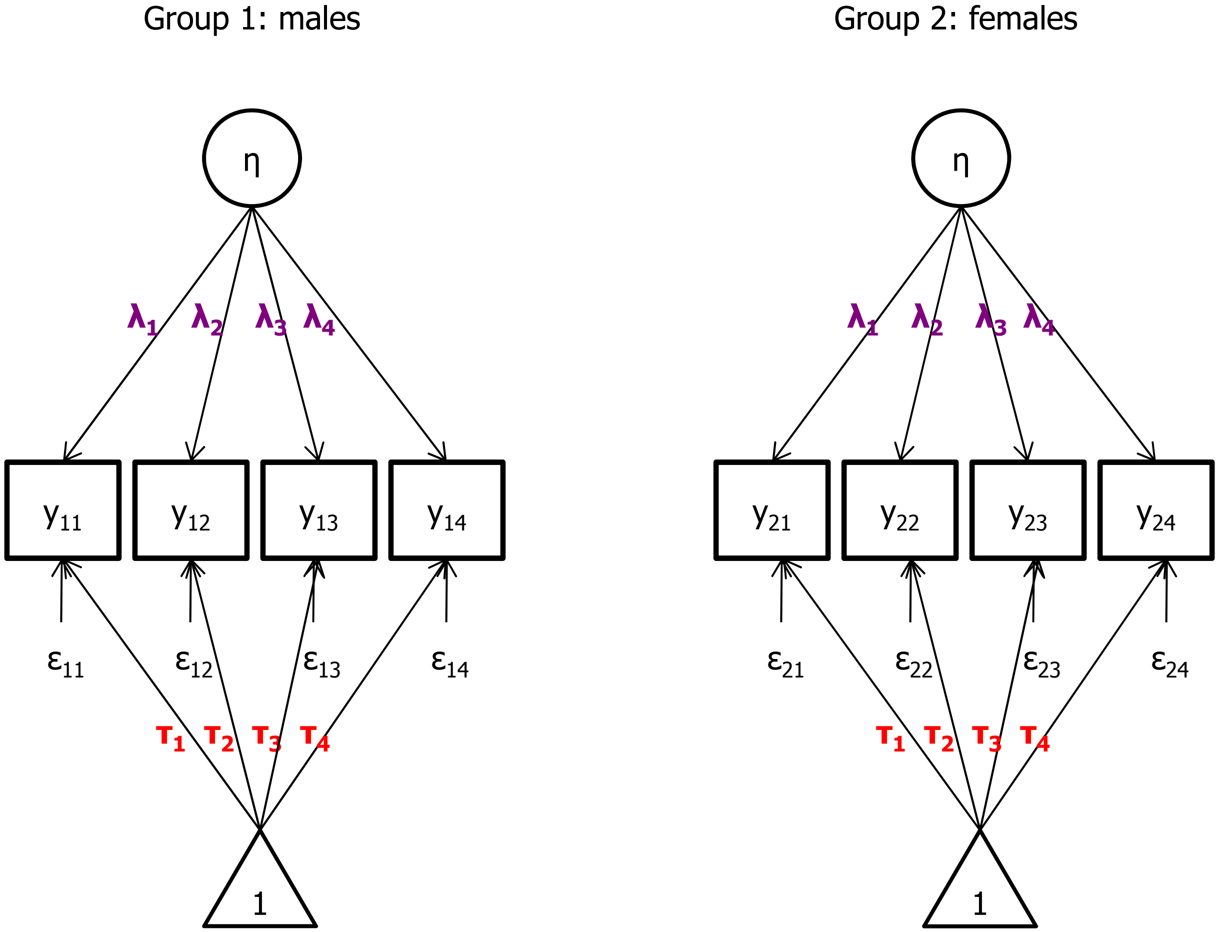
Establishing MI: testing a number of increasingly restrictive models

MODEL 2: Metric invariance -> equal factor loadings over the groups



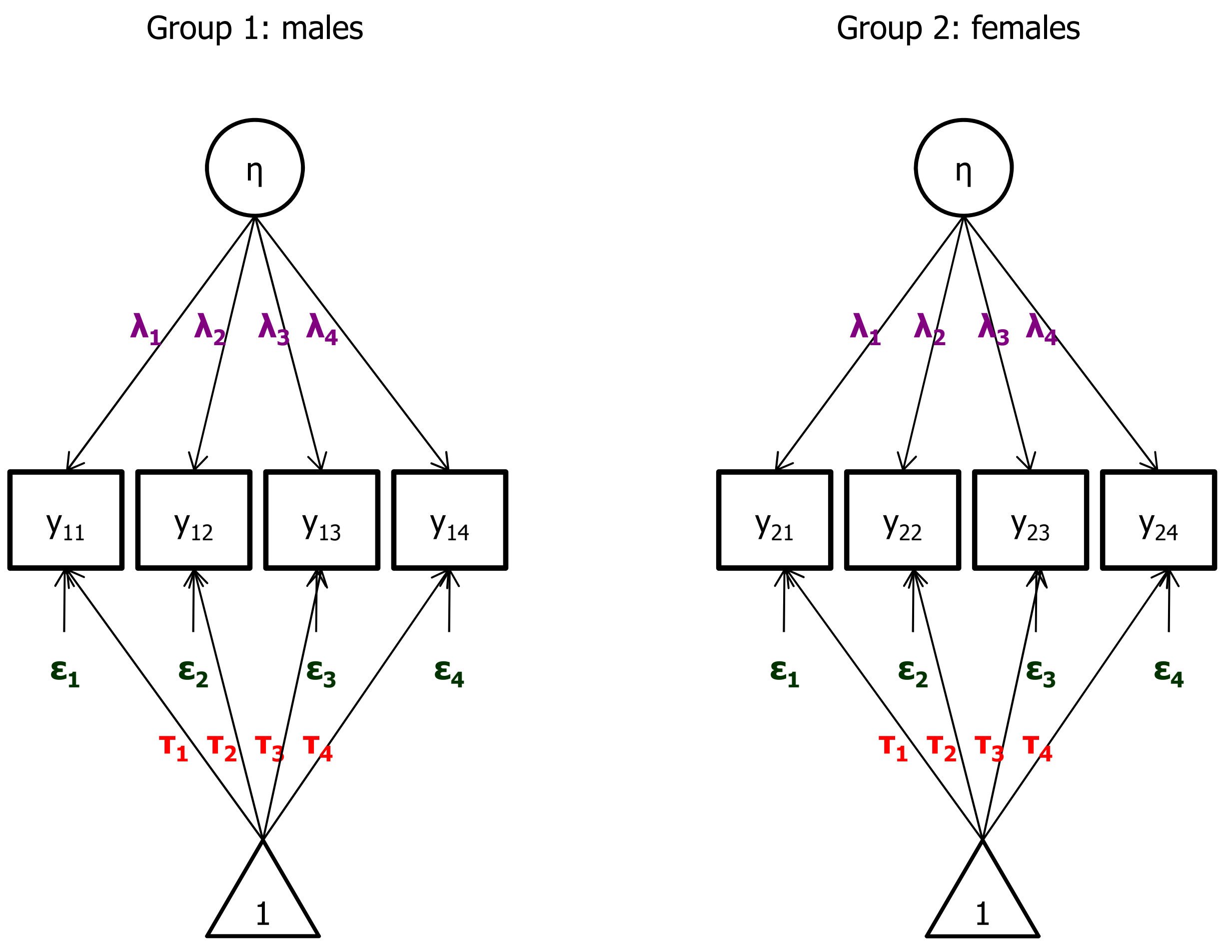
Establishing MI: testing a number of increasingly restrictive models

MODEL 3: Strong factorial invariance -> equal factor loadings and intercepts over the groups



Establishing MI: testing a number of increasingly restrictive models

MODEL 4: Strict factorial invariance -> equal factor loadings, intercepts and residual variances over the groups



Current practical: Are the 4 subscales of the WAIS-III measurement invariant with respect to gender?

Data:

	vci	poi	wmi	psi
gender	**scale1**	**scale2**	**scale3**	**scale4**
2	11	9.33	10.33	13.5
2	10.67	9	10.33	15
2	9.67	7.67	9.33	8.5
2	13	10	8.67	9
2	11	11	13.67	17
2	10	12	9.33	11

...

N = 160 individuals (80 male, 100 female)

Subscales:

vci -- Verbal Comprehension Index
poi -- Perceptual Organization Index
wmi -- Working Memory Index
psi -- Processing Speed Index

Current practical: Are the 4 subscales of the WAIS-III measurement invariant with respect to gender?

OpenMx code:

```
#==================================================================#
#   PREPARE DATA                                                     #
#==================================================================#

nv <- 4 # number of phenotype variables to be analyzed
nf <- 1 # number of common factors in the model
selVars <- paste("scale",1:nv,sep="") # phenotype variables to be analyzed
grVars <- c('gender') # grouping variable

data <- read.table(paste(getwd(),"/Measurement_invariance_data.dat",sep=""),header=TRUE)
mData <- round(data[data$gender==1, selVars],2)
fData <- round(data[data$gender==2, selVars],2)

# Generate descriptive statistics
colMeans(mData,na.rm=TRUE)
colMeans(fData,na.rm=TRUE)
cov(mData,use="complete")
cov(fData,use="complete")

# Test for a mean difference between males and females (MANOVA)
summary(manova(cbind(scale1,scale2,scale3,scale4) ~ gender, data = data), test = "Pillai")
```

Current practical: Are the 4 subscales of the WAIS-III measurement invariant with respect to gender?

OpenMx code:

```
#==========================================================================#
#   PREPARE MODEL                                                          #
#==========================================================================#

# Matrices to store factor loadings of the WAIS subscales on g
loadings1 <- mxMatrix( type="Full", nrow=nv, ncol=nf, free=c(F, rep(T,nv-1)),
values=1, label=paste("l_1", 1:nv, sep=""), name="load1" )
loadings2 <- mxMatrix( type="Full", nrow=nv, ncol=nf, free=c(F, rep(T,nv-1)),
values=1, label=paste("l_2", 1:nv, sep=""), name="load2" )

# Matrices to store the residual variances of the WAIS subscales
residuals1 <- mxMatrix( type="Diag", nrow=nv, free=T, values=2,
label=paste("res_1", 1:nv, sep=""), name="res1" )
residuals2 <- mxMatrix( type="Diag", nrow=nv, free=T, values=2,
label=paste("res_2", 1:nv, sep=""), name="res2" )

# Matrices to store the mean and variance of g (variance estimated, mean set to 0)
latVariance1 <- mxMatrix( type="Symm", nrow=nf, ncol=nf, free=T, values=4,
label=paste("lVar_1", 1:nf, sep=""), name="latVar1" )
latVariance2 <- mxMatrix( type="Symm", nrow=nf, ncol=nf, free=T, values=4,
label=paste("lVar_2", 1:nf, sep=""), name="latVar2" )

latMean1 <- mxMatrix( type="Full", nrow=1, ncol=nf, free=F, values=0,
label=paste("lMean_1",1:nf, sep=""), name="latM1" )
latMean2 <- mxMatrix( type="Full", nrow=1, ncol=nf, free=F, values=0,
label=paste("lMean_2",1:nf, sep=""), name="latM2" )
```

Current practical: Are the 4 subscales of the WAIS-III measurement invariant with respect to gender?

OpenMx code:

```
# Vectors to store intercepts of the WAIS subscales
intercepts1 <- mxMatrix( type="Full", nrow=nv, ncol=1, free=T, values=8,
label=paste("int_1",1:nv,sep=""), name="int1" )
intercepts2 <- mxMatrix( type="Full", nrow=nv, ncol=1, free=T, values=8,
label=paste("int_2",1:nv,sep=""), name="int2" )

# Algebra for the expected means and covariances of the WAIS scores
means1 <- mxAlgebra( expression=t(int1 + load1%*%latM1), name="m1" )
means2 <- mxAlgebra( expression=t(int2 + load2%*%latM2), name="m2" )
variances1 <- mxAlgebra( expression=load1 %*% latVar1 %*% t(load1) + res1, name="v1" )
variances2 <- mxAlgebra( expression=load2 %*% latVar2 %*% t(load2) + res2, name="v2" )

# Data objects for the two groups
data1 <- mxData( observed=mData, type="raw" )
data2 <- mxData( observed=fData, type="raw" )

# Objective objects for the two groups
obj1 <- mxFIMLObjective( covariance="v1", means="m1", dimnames=selVars )
obj2 <- mxFIMLObjective( covariance="v2", means="m2", dimnames=selVars )

# Combine Groups
modelMales  <- mxModel( loadings1, residuals1, latVariance1, latMean1,
intercepts1, means1, variances1, data1, obj1, name="males")
modelFemales  <- mxModel( loadings2, residuals2, latVariance2, latMean2,
intercepts2, means2, variances2, data2, obj2, name="females")
minus2ll <- mxAlgebra( expression=males.objective + females.objective, name="m2LL" )
obj      <- mxAlgebraObjective( "m2LL" )
CImodel  <- mxModel( "CI", modelMales, modelFemales, minus2ll, obj )
```

Current practical: Are the 4 subscales of the WAIS-III measurement invariant with respect to gender?

OpenMx code:

```
#===========================================================================#
#   RUN MODEL: CONFIGURAL INVARIANCE                                        #
#              - equal configuration of factor loadings over the groups     #
#===========================================================================#

CImodelFit    <- mxRun(CImodel)
CImodelSumm   <- summary(CImodelFit)
CImodelSumm
```

Current practical: Are the 4 subscales of the WAIS-III measurement invariant with respect to gender?

OpenMx code:

```
#=======================================================================#
#   RUN MODEL: METRIC INVARIANCE                                        #
#            - equal configuration of factor loadings over the groups   #
#            - equal factor loadings over the groups                    #
#=======================================================================#

# Matrices to store factor loadings of the WAIS subscales on g
loadings1 <- mxMatrix( type="Full", nrow=nv, ncol=nf, free=c(F, rep(T,nv-1)),
values=1, label=paste("l_", 1:nv, sep=""), name="load1" )
loadings2 <- mxMatrix( type="Full", nrow=nv, ncol=nf, free=c(F, rep(T,nv-1)),
values=1, label=paste("l_", 1:nv, sep=""), name="load2" )

# Combine Groups
modelMales  <- mxModel( loadings1, residuals1, latVariance1, latMean1,
intercepts1, means1, variances1, data1, obj1, name="males")
modelFemales  <- mxModel( loadings2, residuals2, latVariance2, latMean2,
intercepts2, means2, variances2, data2, obj2, name="females")
minus2ll <- mxAlgebra( expression=males.objective + females.objective, name="m2LL" )
obj      <- mxAlgebraObjective( "m2LL" )
MImodel  <- mxModel( "MI", modelMales, modelFemales, minus2ll, obj )

MImodelFit    <- mxRun(MImodel)
MImodelSumm   <- summary(MImodelFit)
MImodelSumm
```

Current practical: Are the 4 subscales of the WAIS-III measurement invariant with respect to gender?

OpenMx code:

```
#==========================================================================#
#   RUN MODEL: STRONG FACTORIAL INVARIANCE - YOUR TASK                      #
#             - equal configuration of factor loadings over the groups      #
#             - equal factor loadings over the groups                       #
#             - equal intercepts over the groups                            #
#==========================================================================#

# Vectors to store intercepts of the WAIS subscales
???
???
???
???

# Combine Groups
modelMales  <- mxModel( loadings1, residuals1, latVariance1, latMean1,
intercepts1, means1, variances1, data1, obj1, name="males")
modelFemales  <- mxModel( loadings2, residuals2, latVariance2, latMean2,
intercepts2, means2, variances2, data2, obj2, name="females")
minus2ll <- mxAlgebra( expression=males.objective + females.objective, name="m2LL" )
obj      <- mxAlgebraObjective( "m2LL" )
SFImodel  <- mxModel( "SFI", modelMales, modelFemales, minus2ll, obj )

SFImodelFit    <- mxRun(SFImodel)
SFImodelSumm   <- summary(SFImodelFit)
SFImodelSumm
```

Current practical: Are the 4 subscales of the WAIS-III measurement invariant with respect to gender?

OpenMx code:

```
#==========================================================================#
#   RUN MODEL: STRONG FACTORIAL INVARIANCE - YOUR TASK                     #
#           - equal configuration of factor loadings over the groups       #
#           - equal factor loadings over the groups                        #
#           - equal intercepts over the groups                             #
#==========================================================================#

# Vectors to store intercepts of the WAIS subscales
intercepts1 <- mxMatrix( type="Full", nrow=nv, ncol=1, free=T, values=8,
label=paste("int_",1:nv,sep=""), name="int1" )
intercepts2 <- mxMatrix( type="Full", nrow=nv, ncol=1, free=T, values=8,
label=paste("int_",1:nv,sep=""), name="int2" )

# Combine Groups
modelMales  <- mxModel( loadings1, residuals1, latVariance1, latMean1,
intercepts1, means1, variances1, data1, obj1, name="males")
modelFemales  <- mxModel( loadings2, residuals2, latVariance2, latMean2,
intercepts2, means2, variances2, data2, obj2, name="females")
minus2ll <- mxAlgebra( expression=males.objective + females.objective, name="m2LL" )
obj      <- mxAlgebraObjective( "m2LL" )
SFImodel  <- mxModel( "SFI", modelMales, modelFemales, minus2ll, obj )

SFImodelFit    <- mxRun(SFImodel)
SFImodelSumm   <- summary(SFImodelFit)
SFImodelSumm
```

Current practical: Are the 4 subscales of the WAIS-III measurement invariant with respect to gender?

OpenMx code:

```
#===================================================================#
#   RUN MODEL: STRICT FACTORIAL INVARIANCE - YOUR TASK                #
#           - equal configuration of factor loadings over the groups  #
#           - equal factor loadings over the groups                   #
#           - equal intercepts over the groups                        #
#           - equal residuals over the groups                         #
#===================================================================#

???
???
???
???
???

???
???
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STFImodelFit    <- mxRun(STFImodel)
STFImodelSumm   <- summary(STFImodelFit)
STFImodelSumm
```

Current practical: Are the 4 subscales of the WAIS-III measurement invariant with respect to gender?

OpenMx code:

```
#==================================================================#
#   RUN MODEL: STRICT FACTORIAL INVARIANCE - YOUR TASK              #
#          - equal configuration of factor loadings over the groups  #
#          - equal factor loadings over the groups                   #
#          - equal intercepts over the groups                        #
#          - equal residuals over the groups                         #
#==================================================================#

# Matrices to store the residual variances of the WAIS subscales
residuals1 <- mxMatrix( type="Diag", nrow=nv, free=T, values=2,
label=paste("res_", 1:nv, sep=""), name="res1" )
residuals2 <- mxMatrix( type="Diag", nrow=nv, free=T, values=2,
label=paste("res_", 1:nv, sep=""), name="res2" )

# Combine Groups
modelMales  <- mxModel( loadings1, residuals1, latVariance1, latMean1,
intercepts1, means1, variances1, data1, obj1, name="males")
modelFemales  <- mxModel( loadings2, residuals2, latVariance2, latMean2,
intercepts2, means2, variances2, data2, obj2, name="females")
minus2ll <- mxAlgebra( expression=males.objective + females.objective, name="m2LL" )
obj      <- mxAlgebraObjective( "m2LL" )
STFImodel  <- mxModel( "STFI", modelMales, modelFemales, minus2ll, obj )

STFImodelFit    <- mxRun(STFImodel)
STFImodelSumm   <- summary(STFImodelFit)
STFImodelSumm
```

Current practical: Are the 4 subscales of the WAIS-III measurement invariant with respect to gender?

OpenMx code:

```
#=======================================================================#
#   RUN BASELINE MODEL: 2-GROUP SATURATED MODEL                          #
#=======================================================================#

# Matrix to store variances/covariances
startCov=cov(data[,selVars])
covariances1 <- mxMatrix( type="Symm", nrow=nv, ncol=nv, free=T,
values=startCov, name="covs1" )
covariances2 <- mxMatrix( type="Symm", nrow=nv, ncol=nv, free=T,
values=startCov, name="covs2" )

# Vector to store the means
means1 <- mxMatrix( type="Full", nrow=1, ncol=4, free=T, values=8,
labels=paste("mean_2",1:nv,sep=""), name="m1" )
means2 <- mxMatrix( type="Full", nrow=1, ncol=4, free=T, values=8,
labels=paste("mean_1",1:nv,sep=""), name="m2" )

# Data object
Data1 <- mxData( observed=mData[,selVars], type="raw" )
Data2 <- mxData( observed=fData[,selVars], type="raw" )

# Objective object
obj1 <- mxFIMLObjective( covariance="covs1", means="m1", dimnames=selVars )
obj2 <- mxFIMLObjective( covariance="covs2", means="m2", dimnames=selVars )
```

Current practical: Are the 4 subscales of the WAIS-III measurement invariant with respect to gender?

OpenMx code:

```
# Combine the groups
satModelMales  <- mxModel( covariances1, means1, Data1, obj1, name="satMales")
satModelFemales  <- mxModel( covariances2, means2, Data2, obj2, name="satFemales")
minus2ll <- mxAlgebra( expression=satMales.objective + satFemales.objective, name="m2LL" )
obj      <- mxAlgebraObjective( "m2LL" )
satModel  <- mxModel( "CI", satModelMales, satModelFemales, minus2ll, obj )

# Run the model
satFit <- mxRun(satModel)
satSumm <- summary(satFit)
satSumm
```

Current practical: Are the 4 subscales of the WAIS-III measurement invariant with respect to gender?

OpenMx code:

```
#=========================================================================#
#   COMPARE MODEL FIT                                                     #
#=========================================================================#

tableFitStatistics(satFit,CImodelFit) # test of configural invariance
tableFitStatistics(CImodelFit,MImodelFit) # test of metric invariance
tableFitStatistics(MImodelFit,SFImodelFit) # test of strong f. invariance
tableFitStatistics(SFImodelFit,STFImodelFit) # test of strict f. invariance
```

Current practical: Are the 4 subscales of the WAIS-III measurement invariant with respect to gender?

OpenMx code:

```
#============================================================================#
#   COMPARE MODEL FIT                                                        #
#============================================================================#

tableFitStatistics(satFit,CImodelFit) # test of configural invariance
tableFitStatistics(CImodelFit,MImodelFit) # test of metric invariance
tableFitStatistics(MImodelFit,SFImodelFit) # test of strong f. invariance
tableFitStatistics(SFImodelFit,STFImodelFit) # test of strict f. invariance
```

Conclusion...?