Matrix Algebra Variation and Likelihood

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Overview

- Matrix multiplication review
- Variation & covariation
- Likelihood Theory
- Practical

Three Kinds of Matrix Multiplication

- Dot/Hadamard: element by element
  - In R use *
  - ncol(A) = ncol(B) AND nrow(A) = nrow(B)
- Ordinary matrix multiplication
  - In R use %*%
  - # of columns in A = # of rows in B
- Kronecker multiplication
  - In R use %x%
  - No conformability conditions
- http://en.wikipedia.org/wiki/Matrix_multiplication#Ordinary_matrix_product

Dot or Hadamard Multiplication

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 \times 1 & 6 \times 2 \\ 7 \times 3 & 8 \times 4 \end{bmatrix}$$

$$\underline{A}_{i,j} \times \underline{B}_{i,j} = \underline{C}_{i,j}$$

Ordinary Multiplication

- Multiply: A (m x n) by B (n by p)

$$C_{ij} = \sum_{k=1}^{n} A_{ik} \times B_{kj}$$

Ordinary Multiplication I

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$$\underline{A} \times \underline{B} =$$

Ordinary Multiplication II

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (5x1) & \\ & \end{bmatrix}$$

$$\underline{A} \times \underline{B} = \underline{C}$$

$$C_{11} = \sum_{k=1}^{n} A_{11} \times B_{11}$$

Ordinary Multiplication III

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (5\times1)+(6\times3) & \\ & \end{bmatrix}$$

$$\underline{A} \times \underline{B} = \underline{C}$$

$$C_{11} = \sum_{k=2}^{n} A_{12} \times B_{21}$$

Ordinary Multiplication IV

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & (5 \times 2)+(6 \times 4) \\ & \end{bmatrix}$$

$$\underline{A} \times \underline{B} = \underline{C}$$

$$C_{12} = \sum_{k=1}^{n} A_{1k} \times B_{k2}$$

Ordinary Multiplication V

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ (7 \times 1)+(8 \times 3) & \end{bmatrix}$$

$$\underline{A} \times \underline{B} = \underline{C}$$

$$C_{21} = \sum_{k=1}^{n} A_{2k} \times B_{k1}$$

Ordinary Multiplication VI

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & (7 \times 2)+(8 \times 4) \end{bmatrix}$$

$$\underline{A} \times \underline{B} = \underline{C}$$

$$C_{22} = \sum_{k=1}^{n} A_{2k} \times B_{k2}$$

Ordinary Multiplication VII

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

$$\underline{A} \times \underline{B} = \underline{C}$$

m x n   n x p   m x p

Kronecker or Direct Product

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1\cdot 0 & 1\cdot 5 & 2\cdot 0 & 2\cdot 5 \\ 1\cdot 6 & 1\cdot 7 & 2\cdot 6 & 2\cdot 7 \\ 3\cdot 0 & 3\cdot 5 & 4\cdot 0 & 4\cdot 5 \\ 3\cdot 6 & 3\cdot 7 & 4\cdot 6 & 4\cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 5 & 0 & 10 \\ 6 & 7 & 12 & 14 \\ 0 & 15 & 0 & 20 \\ 18 & 21 & 24 & 28 \end{bmatrix}$$

Kronecker or Direct Product

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1\cdot 0 & 1\cdot 5 & 2\cdot 0 & 2\cdot 5 \\ 1\cdot 6 & 1\cdot 7 & 2\cdot 6 & 2\cdot 7 \\ 3\cdot 0 & 3\cdot 5 & 4\cdot 0 & 4\cdot 5 \\ 3\cdot 6 & 3\cdot 7 & 4\cdot 6 & 4\cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 5 & 0 & 10 \\ 6 & 7 & 12 & 14 \\ 0 & 15 & 0 & 20 \\ 18 & 21 & 24 & 28 \end{bmatrix}$$

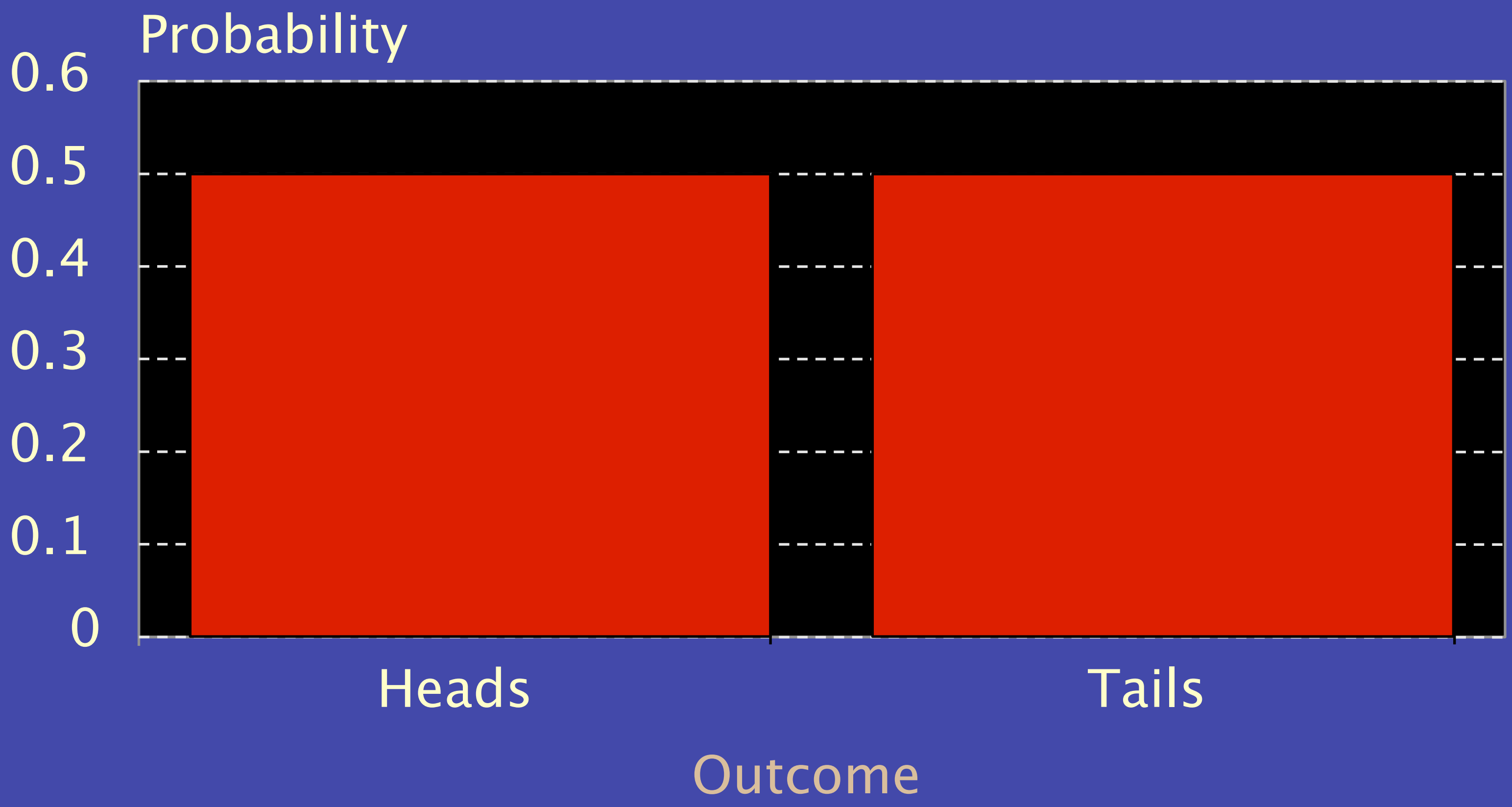
Summarizing Variation and Likelihood

Computing Mean

- Formula $\sum(x_i)/N$
- Can compute with
  - Pencil
  - Calculator
  - SAS
  - SPSS
  - R mean(dataframe)
  - OpenMx

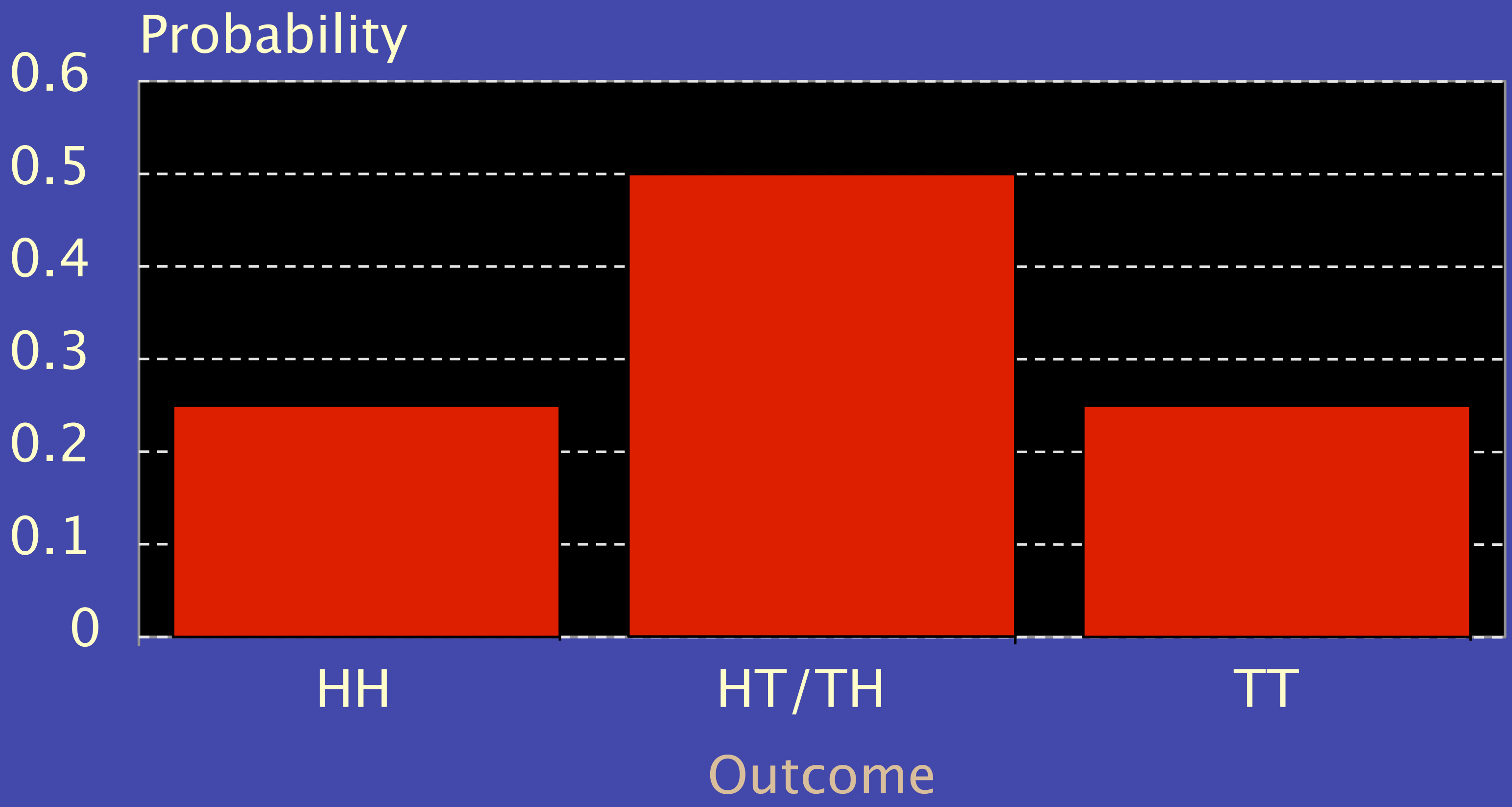
Variance: elementary probability theory

2 outcomes



Two Coin toss

3 outcomes



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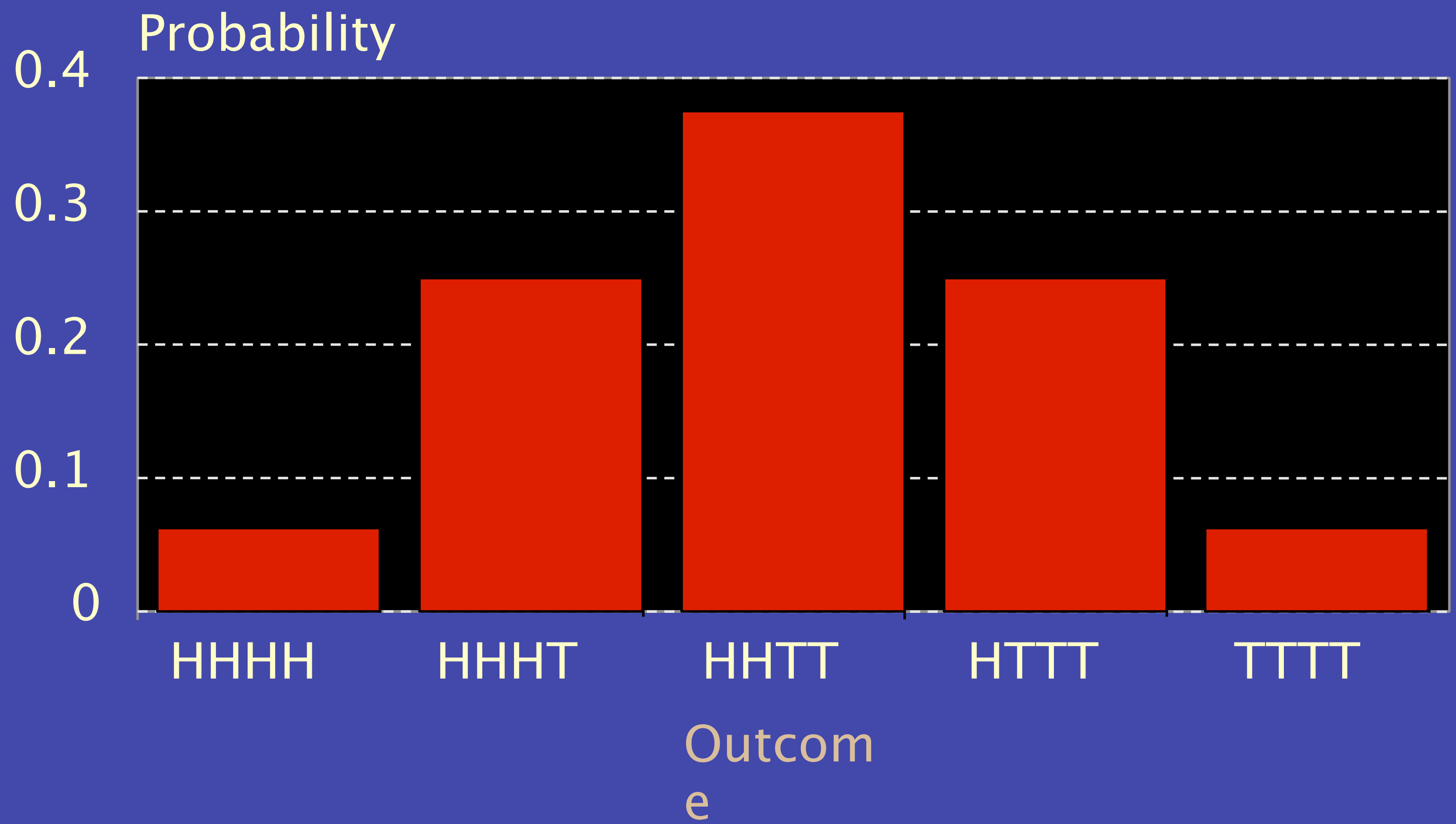
Four Coin toss

5 outcomes



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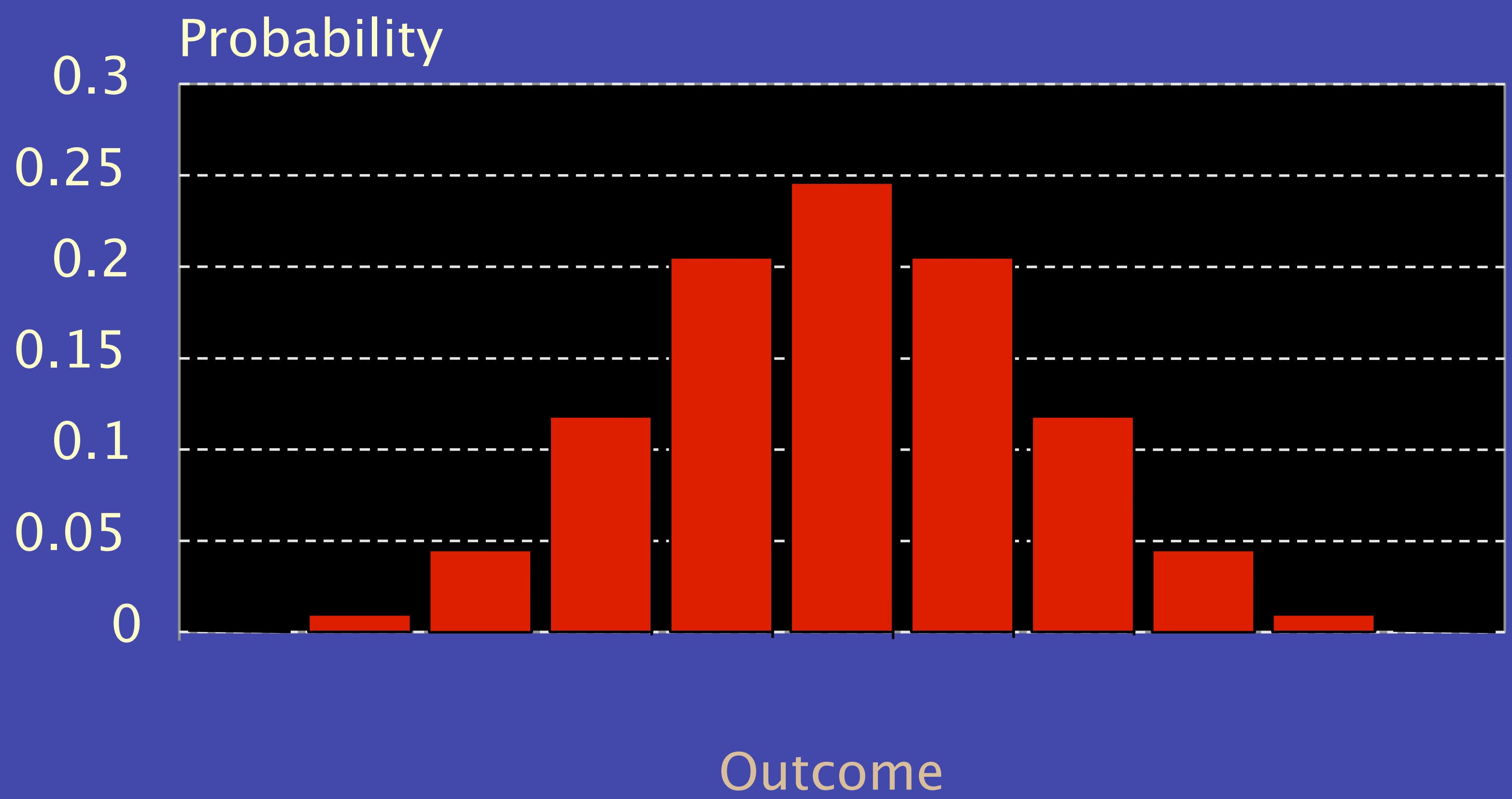
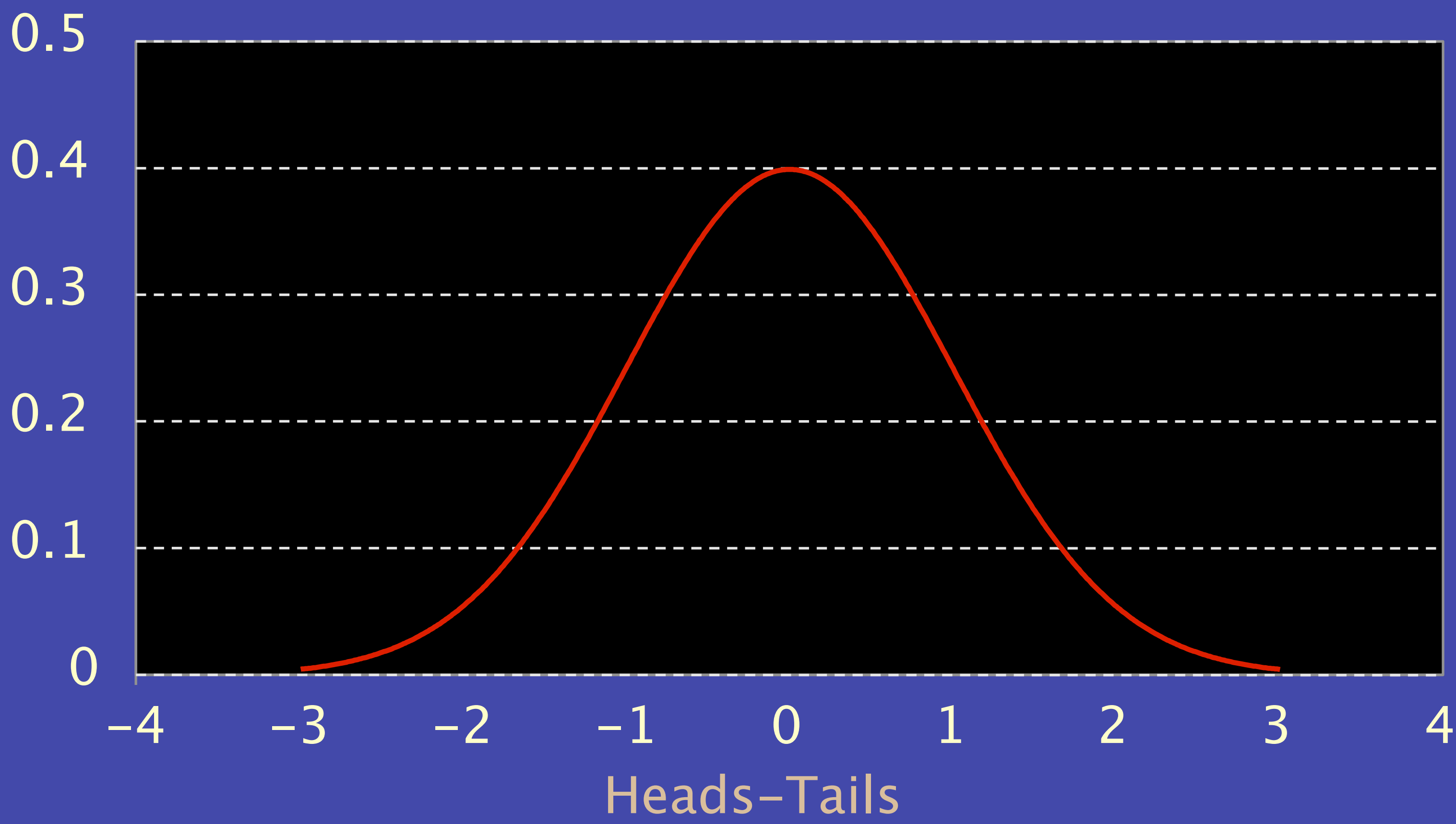
Ten Coin toss

9 outcomes



Fort Knox Toss

Infinite outcomes



De Moivre 1733 Gauss 1827

Dinosaur (of a) Joke

- Elk:
  The Theory by A. Elk brackets Miss brackets. My theory is along the following lines.

- Host:
  Oh God.

- Elk:
  All brontosauruses are thin at one end, much MUCH thicker in the middle, and then thin again at the far end.

http://www.youtube.com/watch?v=cAYDiPizDIs

Pascal's Triangle

Frequency	Probability
1	1/1
1 1	1/2
1 2 1	1/4
1 3 3 1	1/8
1 4 6 4 1	1/16
1 5 10 10 5 1	1/32
1 6 15 20 15 6 1	1/64
1 7 21 35 35 21 7 1	1/128

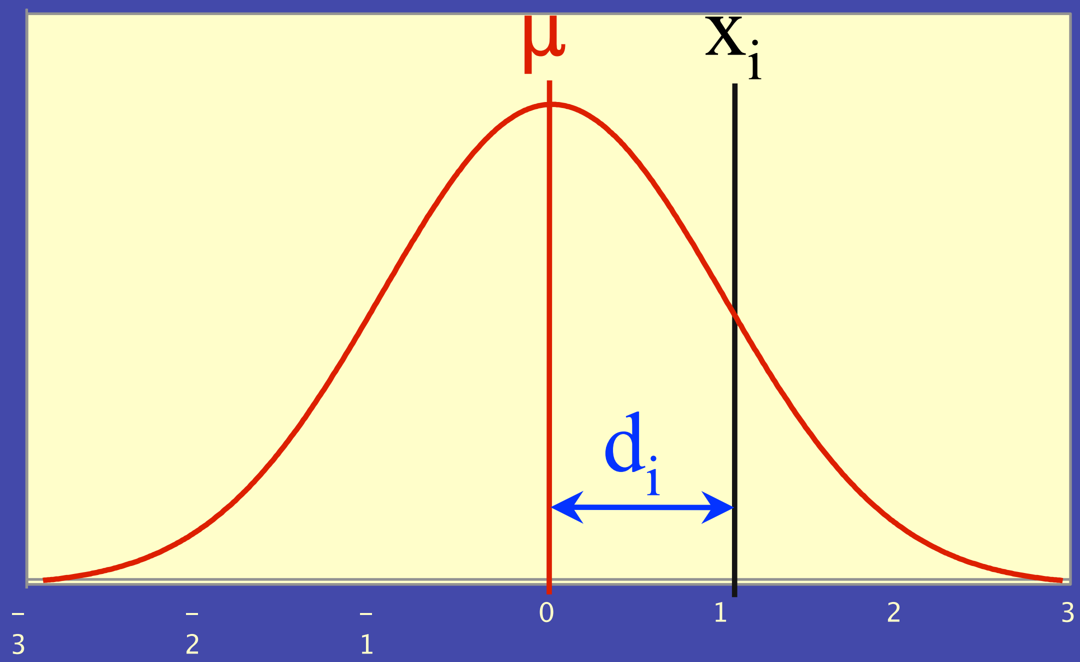
Pascal's friend Chevalier de Mere 1654; Huygens 1657; Cardan 1501-1576

Variance

- Measure of Spread
- Easily calculated
- Individual differences

Average squared deviation

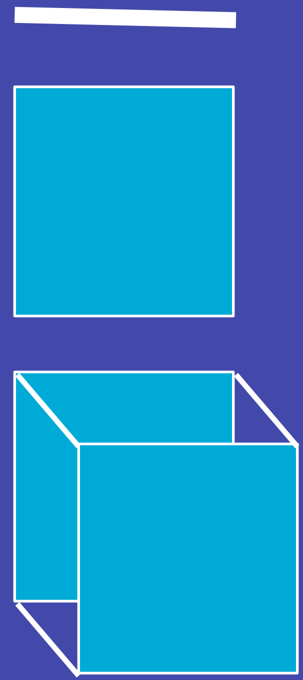
Normal distribution



$$\text{Variance} = \Sigma\ d_i^{\,2}/N$$

Measuring Variation

Weighs & Means



- Absolute differences?
- Squared differences?
- Absolute cubed?
- Squared squared?

?

Measuring Variation

Weighs & Means

- Squared differences

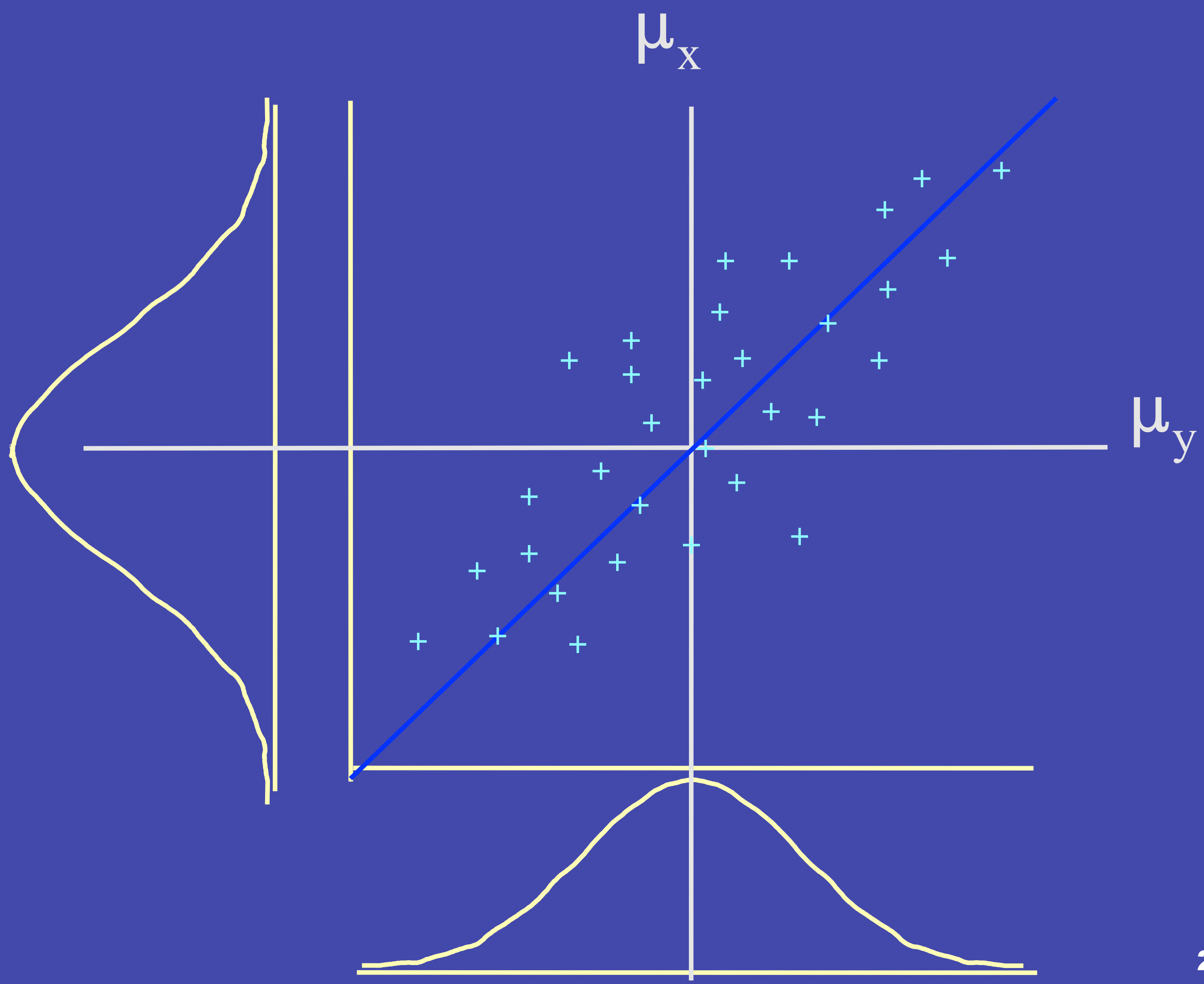
Fisher (1922) On the mathematical foundations of theoretical statistics. *Phil Trans Roy Soc London A* **222**:309-368

Squared has minimum variance under normal distribution

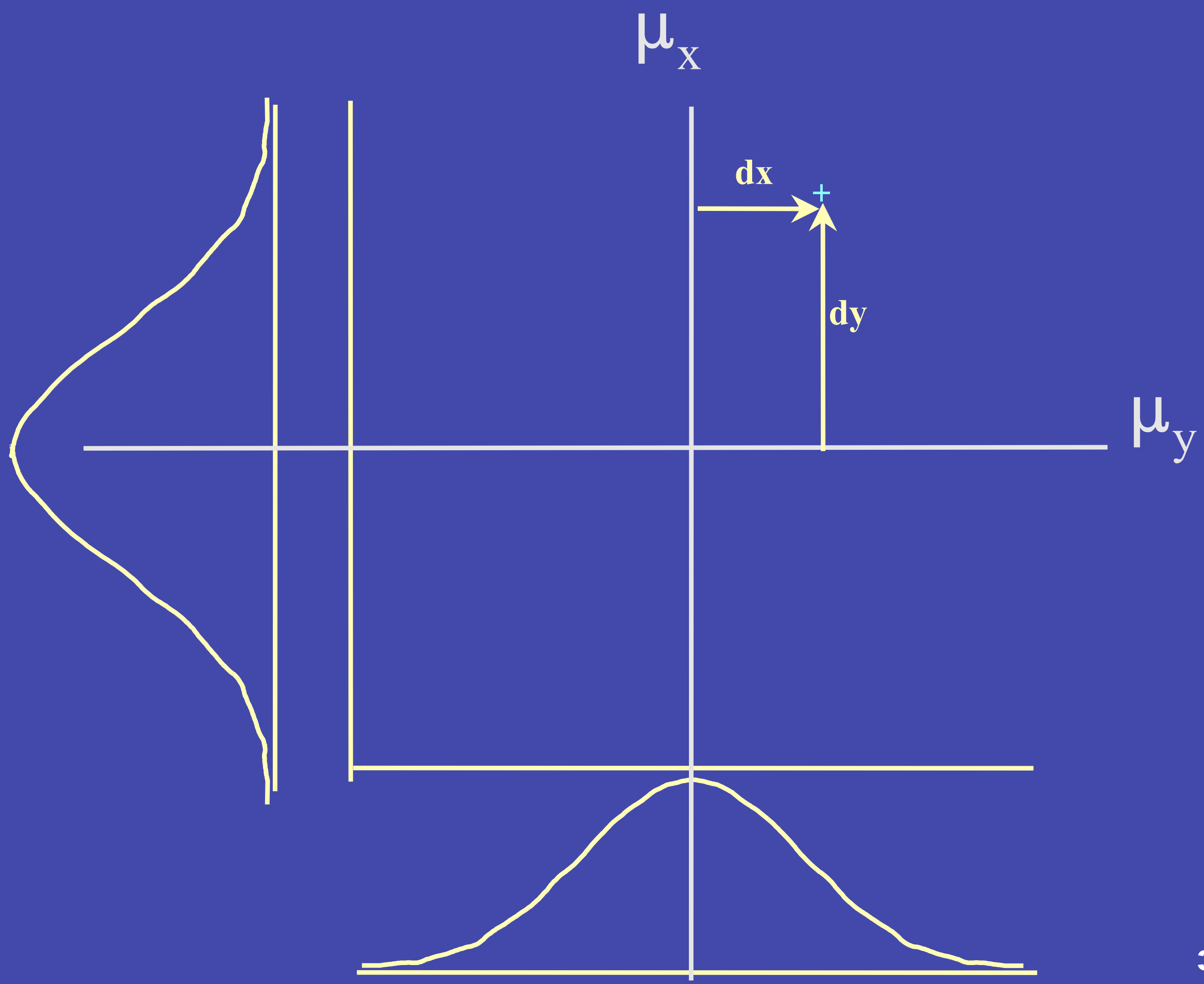
Covariance

- Measure of association between two variables
- Closely related to variance
- Useful to partition variance

Deviations in two dimensions



Deviations in two dimensions



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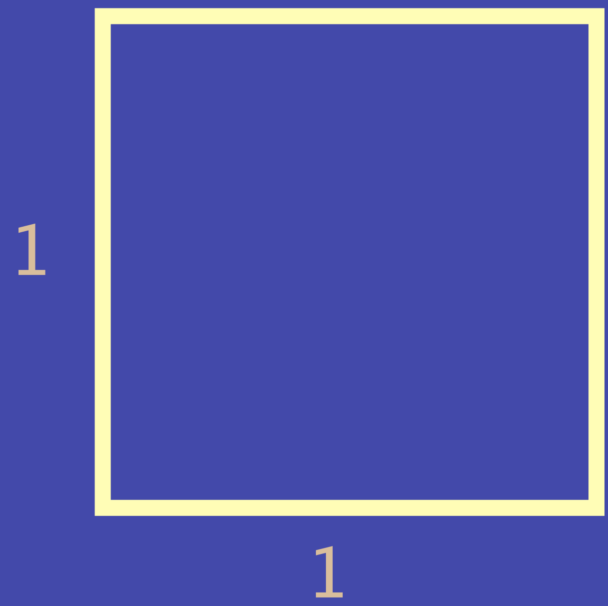
Measuring Covariation

Concept: Area of a rectangle

- A square, perimeter 4
- Area 1



Measuring Covariation

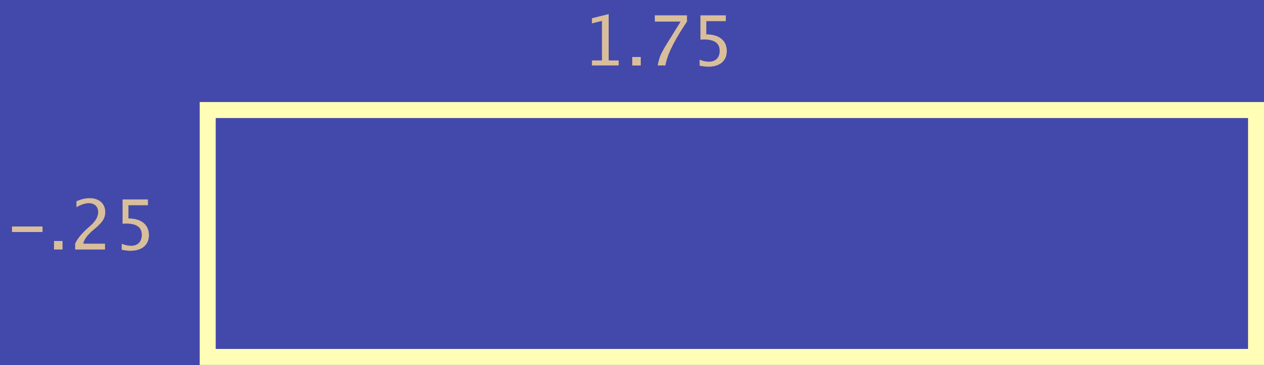
Concept: Area of a rectangle

- A skinny rectangle, perimeter 4
- Area .25*1.75 = .4385

Measuring Covariation

Concept: Area of a rectangle

- Points can contribute negatively
- Area -.25*1.75 = -.4385



Measuring Covariation

Covariance Formula: Average cross-product of deviations from mean

$$\sigma_{xy} = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{N}$$

Correlation

- Standardized covariance
- Lies between -1 and 1

$$r_{xy} = \frac{\sigma_{xy}}{\sqrt{\sigma_x^2 * \sigma_y^2}}$$

Summary

Formulae for sample statistics; i=1…N observations

$$\mu = (\Sigma x_i)/N$$

$$\sigma_x^2 = \Sigma (x_i - \mu_x)^2 / (N)$$

$$\sigma_{xy} = \Sigma (x_i - \mu_x)(y_i - \mu_y) / (N)$$

$$r_{xy} = \frac{\sigma_{xy}}{\sqrt{\sigma_x^2 \quad \sigma_x^2}}$$

Variance covariance matrix

Several variables

$$\begin{bmatrix} Var(X) & Cov(X,Y) & Cov(X,Z) \\ Cov(X,Y) & Var(Y) & Cov(Y,Z) \\ Cov(X,Z) & Cov(Y,Z) & Var(Z) \end{bmatrix}$$

Variance covariance matrix

Univariate Twin Data

$$\begin{bmatrix} \text{Var(Twin1)} & \text{Cov(Twin1,Twin2)} \\ \text{Cov(Twin2,Twin1)} & \text{Var(Twin2)} \end{bmatrix}$$

Only suitable for complete data
Good conceptual perspective

Conclusion

- Means and covariances
- Basic input statistics for “Traditional SEM”
- Easy to compute
- Can use raw data instead

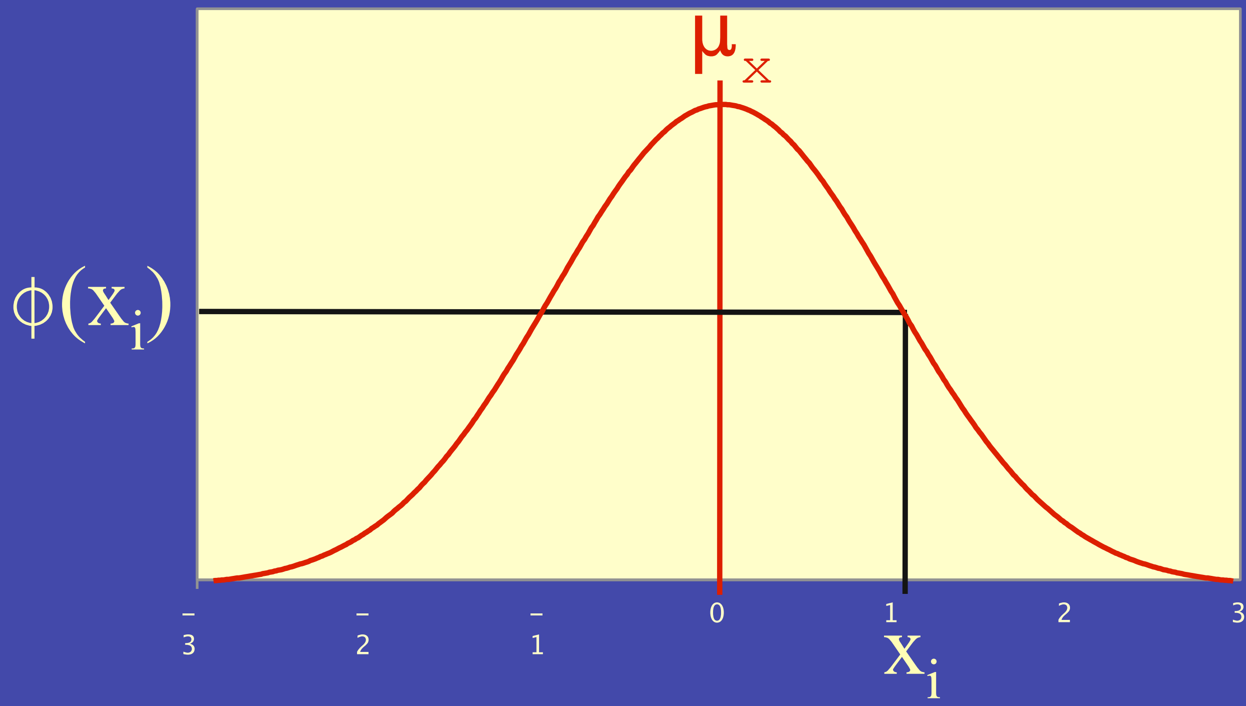
Likelihood: Normal Theory

Calculate height of curve

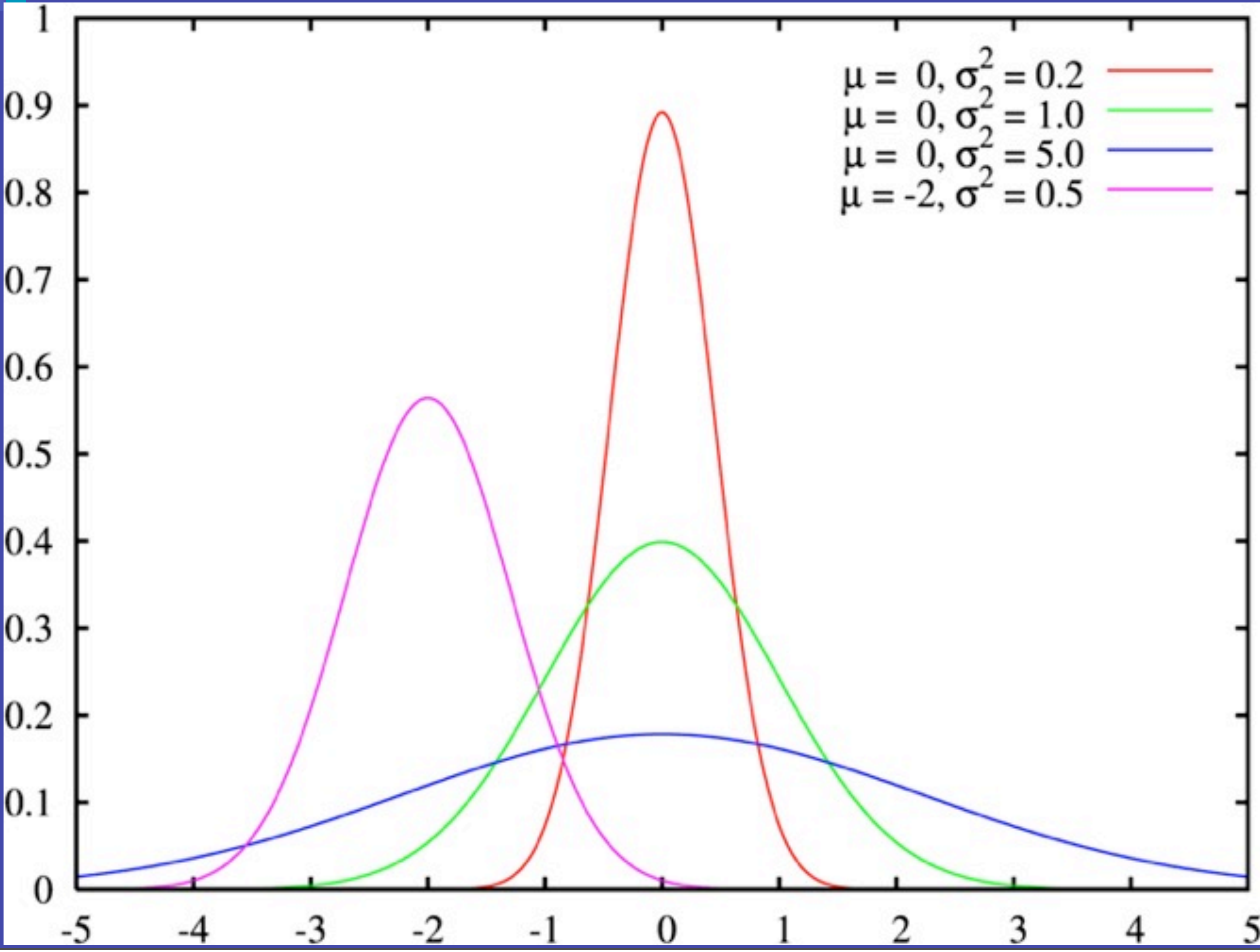
- Univariate - height of normal pdf
  - $\phi(x) =$
  - $(2\Pi\sigma^2)^{-.5}\ e^{-.5((x_i - \mu)\hat{}2)/\sigma\hat{}2)}$
- Multivariate - height of multinormal pdf
  - $|2\Pi\Sigma|^{-n/2}\ e^{-.5((\mathbf{x}_i - \mu)\Sigma^{-1}(\mathbf{x}_i - \mu)')}$

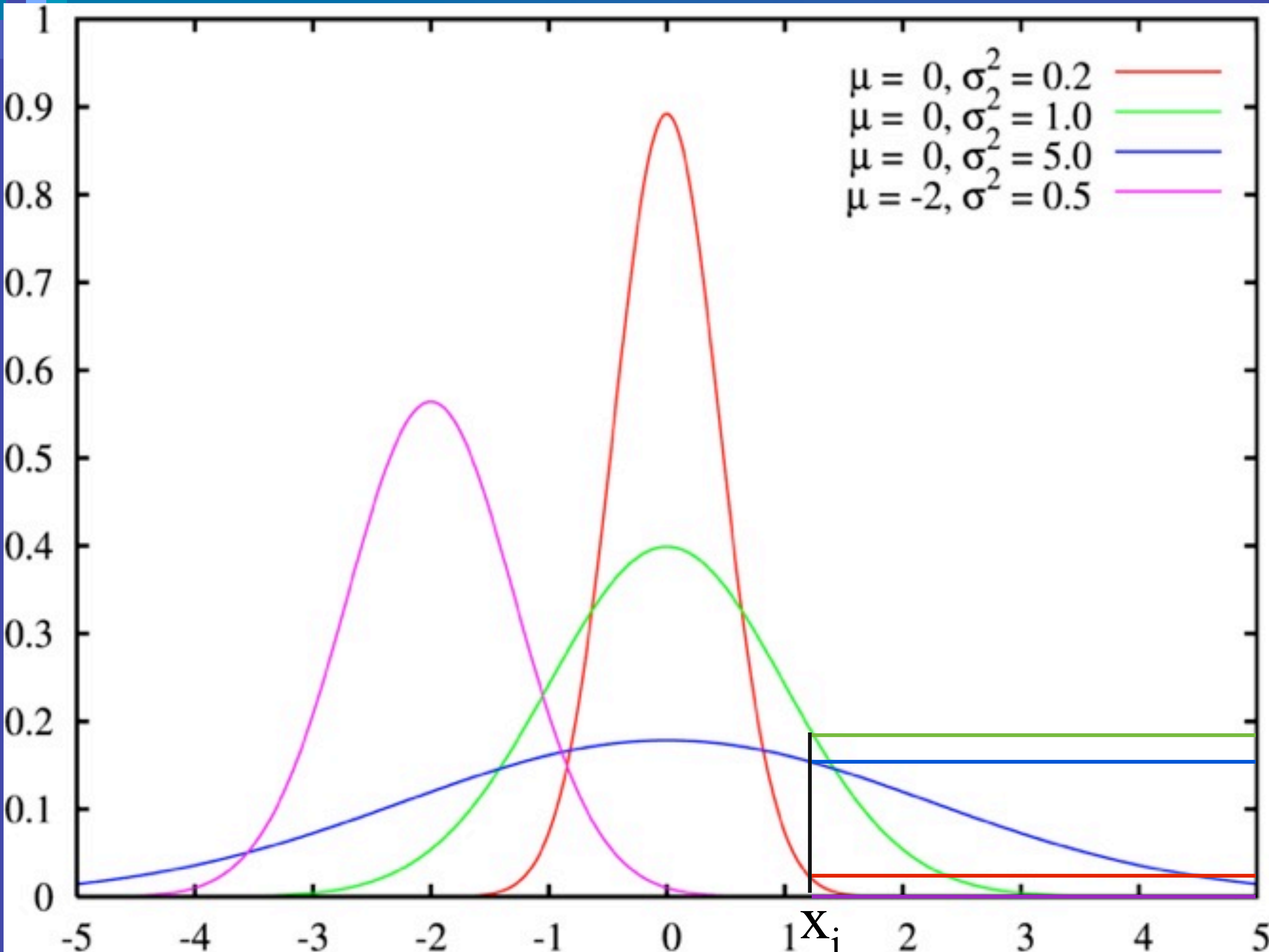
Height of normal curve

Probability density function



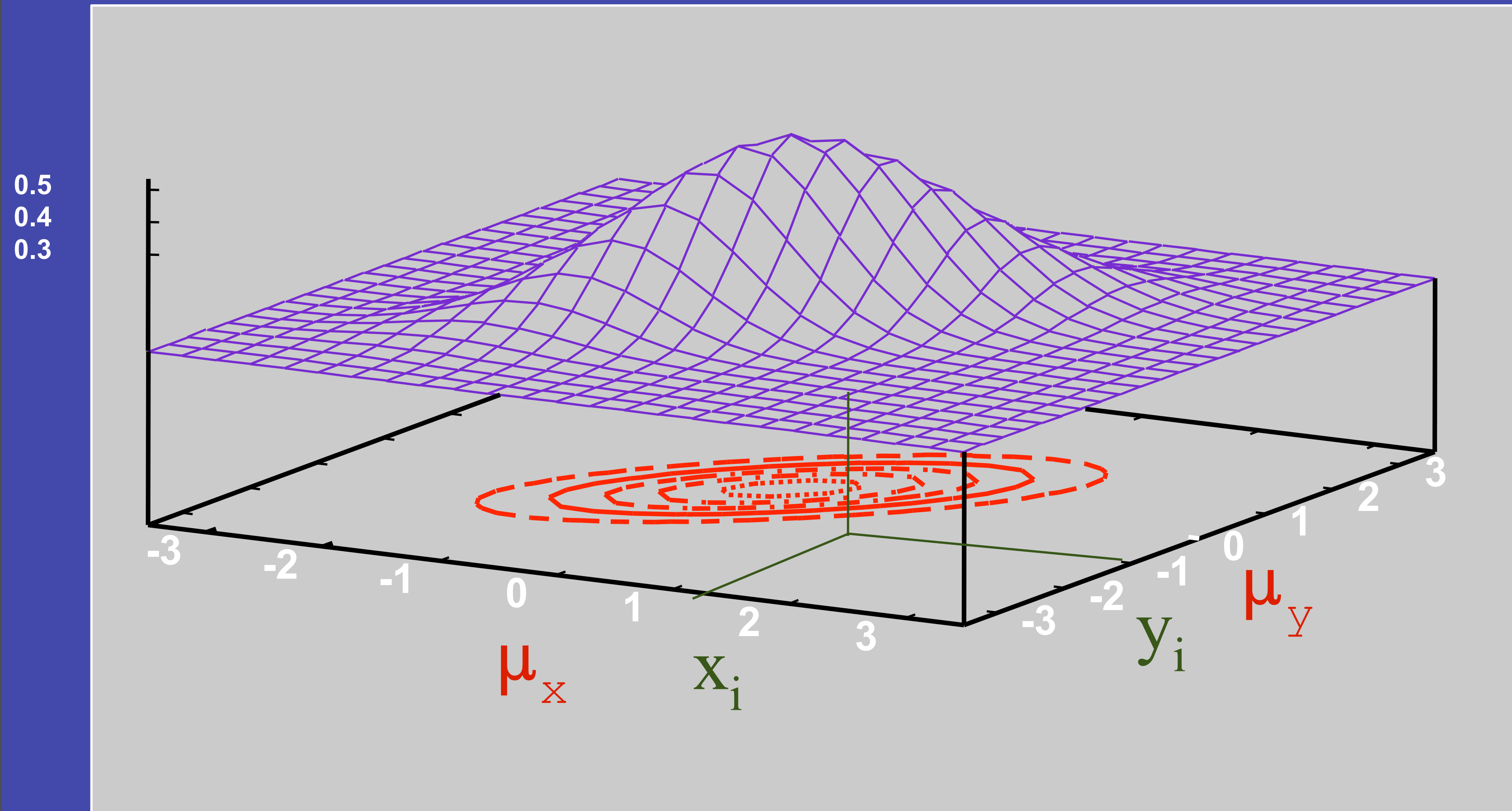
$\phi(x_i)$ is the likelihood of data point x_i for particular mean & variance estimates


μ = 0, σ² = 0.2
μ = 0, σ² = 1.0
μ = 0, σ² = 5.0
μ = -2, σ² = 0.5
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-5
-4
-3
-2
-1
0
1
2
3
4
5


μ = 0, σ² = 0.2
μ = 0, σ² = 1.0
μ = 0, σ² = 5.0
μ = -2, σ² = 0.5
Xi

Height of bivariate normal curve

An unlikely pair of (x,y) values



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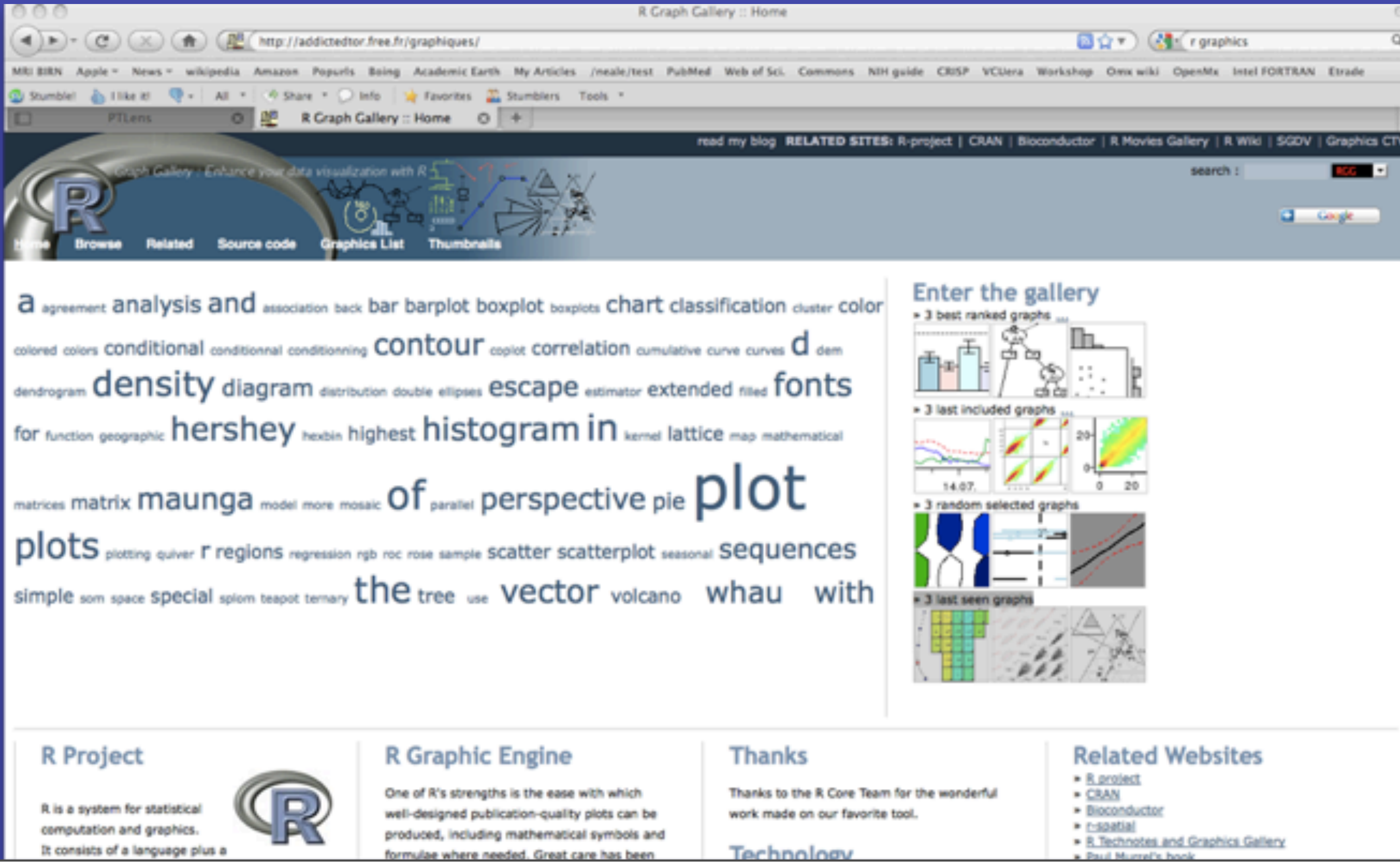
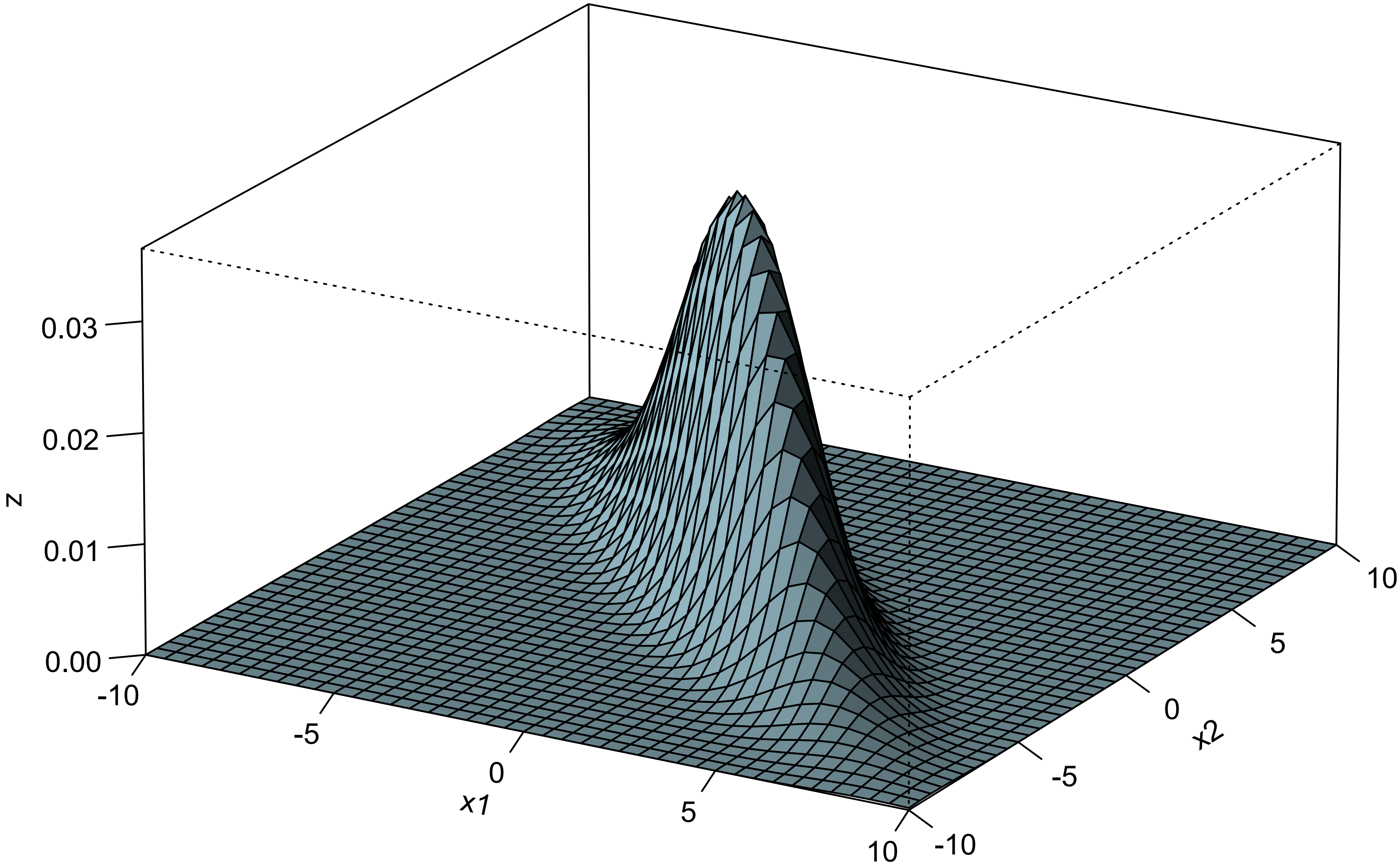
R graphics: http://addictedtor.free.fr/graphiques/



```r
mu1<-0   # set the expected value of x1
mu2<-0   # set the expected value of x2
s11<-10  # set the variance of x1
s22<-10  # set the variance of x2
rho<-0.0 # set the correlation coefficient between x1 and x2
x1<-seq(-10,10,length=41) # generate series for values of x1 & x2
x2<-x1 # copy x1 to x2

# set up the bivariate normal density function - could use
f<-function(x1,x2){
     term1 <- 1/(2*pi*sqrt(s11*s22*(1-rho^2)))
     term2 <- -1/(2*(1-rho^2))
     term3 <- (x1-mu1)^2/s11
     term4 <- (x2-mu2)^2/s22
     term5 <- -2*rho*((x1-mu1)*(x2-mu2))/(sqrt(s11)*sqrt(s22))
     term1*exp(term2*(term3+term4-term5))
}

# calculate the density values
z<-outer(x1,x2,f)

# generate the 3-D plot
persp(x1, x2, z,
      main="Two dimensional Normal Distribution",
      sub=expression(italic(f)~(bold(x))==frac(1,2~pi~sqrt(sigma[11]~
                     sigma[22]~(1-rho^2)))~phantom(0)~exp~bgroup("{",
                  list(-frac(1,2(1-rho^2)),
                  bgroup("[", frac((x[1]~-~mu[1])^2, sigma[11])~-~2~rho~frac(x[1]~-~mu[1],
                  sqrt(sigma[11]))~ frac(x[2]~-~mu[2],sqrt(sigma[22]))~+~
                  frac((x[2]~-~mu[2])^2, sigma[22]),"]")),"}")),
      col="lightblue", theta=30, phi=20, r=50, d=0.1, expand=0.5,
      ltheta=90, lphi=180, shade=0.75, ticktype="detailed", nticks=5)

# adding a text line to the graph
mtext(expression(list(mu[1]==0,mu[2]==0,sigma[11]==10,sigma[22]==10,sigma[12]==15,rho==0.0)),
side=3)
```

Two dimensional Normal Distribution

$\mu_1 = 0, \mu_2 = 0, \sigma_{11} = 10, \sigma_{22} = 10, \sigma_{12} = 15, \rho = 0.9$



$$f(\mathbf{x}) = \frac{1}{2\pi\sqrt{\sigma_{11}\sigma_{22}(1-\rho^2)}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_{11}} - 2\rho\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}} + \frac{(x_2-\mu_2)^2}{\sigma_{22}}\right]\right\}$$

Exercises: Compute Normal PDF

- Get used to OpenMx script language
- Use matrix algebra
- Taste of likelihood theory